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The Bateman–Luke variational formalism in a sloshing with rotational flows

Based on a presentation of the velocity field in terms of Clebsch potentials, the Bateman–Luke variational formalism is generalized for the sloshing of an ideal incompressible liquid with rotational flows.

Keywords: Bateman–Luke variational principle, sloshing, Clebsch potentials.

Using the Bateman–Luke variational formulation [1, 2] is a commonly accepted approach to analytic studies of nonlinear liquid sloshing [3–5]. The formulation suggests an ideal incompressible liquid with an irrotational velocity field that is both consistent with Kelvin's circulation theorem and supported by experiments for a clean (without internal structures) rigid tank. According to Kelvin' and Laplace' results on the invariance of circulation, the *vorticity cannot be generated in the interior of a viscous incompressible fluid subject to a conservative extraneous force, but is necessarily diffused inward from the boundaries*; for clean tanks, the shedding vortices remain localized near the smooth wetted tank surface.

Referring to his friend, Carl Runge, Ludwig Prandtl [6] writes that Pierre-Simon marquis de Laplace demonstrated, in personal communications with colleagues, a "glass–wine" paradox showing that a steady-state swirl sloshing generates, almost immediately, a stable sloshing-supported vortex in the liquid domain by a conversion of the wave angular momentum to the vortex angular momentum. The conversion phenomenon makes inconsistent the existing mathematical theories of circulation, since the vortex is neither appears at nor diffuses from the tank walls. Prandtl [6] conducted a dedicated model test confirming the paradox. Analogous experimental observations were reported in [7, 8], where the authors attempted to explain the paradox, without a serious success, by the angular Stokes drift [3, Sect. 9.6.3], which together with the wave breaking could be a trigger of the conversion, but not its driver. Studying the phenomenon requires a revision of the existing analytic methods to include the rotational solenoidal flows into account. Using Clebsch potentials [9, 10] and ideas by Bateman [1, p. 164–166], the present paper generalizes the Bateman–Luke variational formalism to a sloshing with rotational flows.

A mobile rigid tank is considered partly filled with an inviscid incompressible liquid (the mass density $\rho = \text{const}$). Figure 1 shows the liquid domain $Q(t)$ bounded by the free surface $\Sigma(t)$ and the wetted tank surface $S(t)$, an absolute (inertial) coordinate system $O'x_1'x_2'x_3'$, and a non-inertial (tank-fixed) coordinate system $Ox_1x_2x_3$. The $Ox_1x_2x_3$-system moves (relatively to $O'x_1'x_2'x_3'$) with the absolute translatory velocity $\boldsymbol{v}_O(t)$ and the instant angular velocity $\boldsymbol{\omega}(t)$, so that any fixed point in the $Ox_1x_2x_3$ has the absolute velocity

$$\boldsymbol{v}_b = \boldsymbol{v}_O + \boldsymbol{\omega} \times \boldsymbol{r}, \tag{1}$$

where $\boldsymbol{r} = (x_1, x_2, x_3)$ is the tank-fixed radius-vector. The gravity potential can be written as

$$U(x_1, x_2, x_3, t) = -\boldsymbol{g} \cdot \boldsymbol{r}', \qquad \boldsymbol{r}' = \boldsymbol{r}_O' + \boldsymbol{r},$$

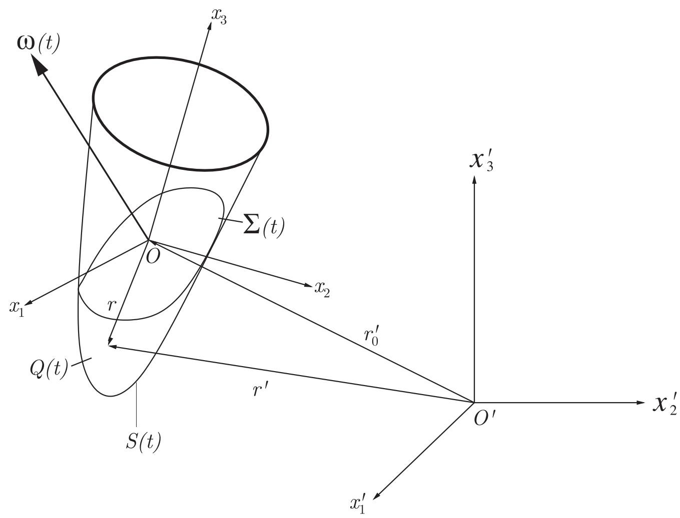

Fig. 1

where $\boldsymbol{r'}$ is the radius-vector of a point of the body–liquid system with respect to O', $\boldsymbol{r}'_O$ is the radius-vector of O with respect to O', and $\boldsymbol{g}$ is the gravity acceleration vector. The free surface $\Sigma(t)$ is implicitly defined in the tank-fixed coordinate system by the equation $Z(x_1, x_2, x_3, t) = 0$ so that the outer normal $\boldsymbol{n}$ to $\Sigma(t)$ is $-\nabla Z/|\nabla Z|$. The function Z is the unknown that satisfies the volume (mass) conservation condition

$$\int\limits_{Q(t)} dQ = V_l = \text{const} \tag{2}$$

treated as a geometric constraint.

The liquid motions are described by three Clebsch potentials $\varphi(x_1, x_2, x_3, t)$, $m(x_1, x_2, x_3, t)$, and $\phi(x_1, x_2, x_3, t)$, so that the *absolute velocity field* $\boldsymbol{v} = (v_1, v_2, v_3, t)$ reads

$$\boldsymbol{v} = \nabla\varphi + m\nabla\phi. \tag{3}$$

Even though relation (3) does not give a unique representation of the velocity field (substitution $m := Cm$, $\phi := \phi/C$, where C is a non-zero constant, confirms that), the Clebsch potentials are henceforth assumed being three independent functions. The case of irrotational flows implies either $m = 0$ or $\phi = \text{const}$.

As remarked in [3, p. 47], the spatial derivatives in the introduced inertial (∂_i') and non-inertial (∂_i) coordinate systems remain the same, but the time-derivatives (∂_t' and ∂_t, respectively) change, i. e.,

$$\partial_i' = \partial_i; \qquad \partial_t' = \partial_t - \boldsymbol{v}_b \cdot \nabla; \qquad d_t' = \partial_t' + \boldsymbol{v} \cdot \nabla = \partial_t + (\boldsymbol{v} - \boldsymbol{v}_b) \cdot \nabla. \tag{4}$$

Based on relations (4) and [1, p. 164], the Lagrangian

$$L(\varphi, m, \phi, Z) = \int\limits_{Q(t)} P\, dQ = -\rho \int\limits_{Q(t)} \left[\partial_t'\varphi + m\partial_t'\phi + \frac{1}{2}|\boldsymbol{v}|^2 + U\right] dQ =$$

$$= -\rho \int\limits_{Q(t)} \left[\partial_t\varphi + m\partial_t\phi - \boldsymbol{v}_b \cdot \boldsymbol{v} + \frac{1}{2}|\boldsymbol{v}|^2 + U\right] dQ \tag{5}$$

and the action

$$W(\varphi, m, \phi, Z) = \int_{t_1}^{t_2} \left[L - p_0 \int_{Q(t)} dQ \right] dt = \int_{t_1}^{t_2} \int_{Q(t)} (P - p_0)\, dQ\, dt \tag{6}$$

are introduced for any fixed instant times $t_1 < t_2$. Functional (6) acts on the independent Clebsch potentials and Z. The Lagrange multiplier p_0 is a consequence of the volume conservation constraint (2).

Henceforth, the assumption is that the Clebsch potentials are smooth functions in $Q(t)$, which admit, for any instant time t, an analytic continuation through the smooth (provided by the admissible Z) free surface $\Sigma(t)$. Using the calculus of variables, Reynolds' transport and divergence theorem makes it possible to prove the following three lemmas.

Lemma 1. *Under the assumption on the smoothness of the Clebsch potentials and the free surface $\Sigma(t)$, the zero first variation*

$$\delta_\varphi W = 0 \qquad \textit{subject to} \qquad \delta\varphi|_{t=t_1,t_2} = 0 \tag{7}$$

is equivalent to the kinematic relations of the sloshing problem consisting of the continuity equation

$$\nabla \cdot (\boldsymbol{v} - \boldsymbol{v}_b) \equiv \nabla \cdot \boldsymbol{v} = 0 \qquad \textit{in} \qquad Q(t), \tag{8}$$

as well as the kinematic boundary conditions

$$(\boldsymbol{v} - \boldsymbol{v}_b) \cdot \boldsymbol{n} = 0 \qquad \textit{on} \qquad S(t), \qquad \textit{and} \qquad (\boldsymbol{v} - \boldsymbol{v}_b) \cdot \boldsymbol{n} = -\frac{\partial_t Z}{|\nabla Z|} \qquad \textit{on} \qquad \Sigma(t), \tag{9}$$

expressing that the normal velocity is defined by the rigid wall motions and the fluid particles remain on the free surface $\Sigma(t)$.

Lemma 2. *Under the assumption on the smoothness of the Clebsch potentials and the free surface $\Sigma(t)$, the zero first variation*

$$\delta_m W = 0 \tag{10}$$

is equivalent to the equation

$$d'\phi \equiv \partial_t'\phi + \boldsymbol{v} \cdot \nabla\phi \equiv \partial_t\phi + (\boldsymbol{v} - \boldsymbol{v}_b) \cdot \nabla\phi = 0 \qquad \textit{in} \qquad Q(t), \tag{11}$$

which indicates that the Clebsch potential ϕ remains constant during the motions of a liquid particle (a vortex line moves with the liquid and always contains the same particles).

Lemma 3. *Under the assumption on the smoothness of the Clebsch potentials and the free surface $\Sigma(t)$, the zero first variation*

$$\delta_\phi W = 0 \qquad \textit{subject to} \qquad \delta\phi|_{t_1,t_2} = 0, \tag{12}$$

and the kinematic problem (8), (9) is equivalent to

$$d'm \equiv \partial_t' m + \boldsymbol{v} \cdot \nabla m \equiv \partial_t m + (\boldsymbol{v} - \boldsymbol{v}_b) \cdot \nabla m = 0 \qquad \textit{in} \qquad Q(t), \tag{13}$$

which has the same meaning that (11), but for the Clebsch potential m.

Remark 1. In contrast to the Bateman–Luke formulation for potential liquid flows, the function P adopted in the definition of Lagrangian (5) is, generally speaking, not the pressure and cannot be treated as the pressure for arbitrary Clebsch potentials. One can show that the pressure $p = P + f(t)$ ($f(t)$ is an arbitrary function), when (11) and (13) are satisfied. In other words, when assuming that (11) and (13) are fulfilled, the Euler equation

$$d'\boldsymbol{v} = -\frac{1}{\rho}(\nabla P + \nabla U) \qquad \text{in} \qquad Q(t) \tag{14}$$

holds true. This fact follows from the expression for the left-hand side of (14)

$$d'(\nabla\varphi + m\nabla\phi) = [\nabla(\partial'_t\varphi) + m\nabla(\partial'_t\phi) + \partial'_t m\nabla\phi] + \underbrace{\boldsymbol{v}\cdot\nabla(\nabla\varphi + m\nabla\phi)}_{\boldsymbol{v}\cdot\nabla\nabla\varphi + m\boldsymbol{v}\cdot\nabla\nabla\phi + \nabla\phi(\nabla m\cdot\boldsymbol{v})} =$$

$$= \nabla(\partial'_t\varphi) + m\nabla(\partial'_t\phi) + \boldsymbol{v}\cdot\nabla\nabla\varphi + m\boldsymbol{v}\cdot\nabla\nabla\phi + \nabla\phi(\nabla m\cdot\boldsymbol{v}) + \nabla\phi[d'm]$$

and the right-hand side (after annihilating the U-term)

$$\nabla\left(\partial'_t\varphi + m\partial'_t\phi + \frac{1}{2}|\boldsymbol{v}|^2\right) = [\nabla(\partial'_t\varphi) + m\nabla(\partial'_t\phi) + \partial'_t\phi\nabla m] + \boldsymbol{v}\cdot\nabla\nabla\varphi + m\boldsymbol{v}\cdot\nabla\nabla\phi +$$

$$+\nabla m(\nabla\phi\cdot\boldsymbol{v}) = \nabla(\partial'_t\varphi) + m\nabla(\partial'_t\phi) + \boldsymbol{v}\cdot\nabla\nabla\varphi + m\boldsymbol{v}\cdot\nabla\nabla\phi + \nabla\phi(\nabla m\cdot\boldsymbol{v}) + \nabla m[d'\phi],$$

in which the framed terms are identical, but the residual terms vanish, as (11) and (13) hold true.

Using Lemmas 1–3 and Remark 1 makes it possible to prove the following theorem.

Theorem 1. *Under the assumption on the smoothness of the Clebsch potentials and the free surface* $\Sigma(t)$, *the zero first variation of action* (6)

$$\delta W = \delta_\varphi W + \delta_m W + \delta_\phi W + \delta_Z W = 0 \tag{15}$$

subject to

$$\delta\varphi|_{t_1,t_2} = \delta\phi|_{t_1,t_2} = 0 \tag{16}$$

is equivalent to the sloshing problem, which includes the kinematic relations (8) *and* (9), *two equations* (11) *and* (13) *expressing the fact that the Clebsch potentials* ϕ *and* m *are constant along the vortex lines, as well as the dynamic boundary condition*

$$p - p_0 = -\rho\left(\partial_t\varphi + m\partial_t\phi - \boldsymbol{v}_b\cdot\boldsymbol{v} + \frac{1}{2}|\boldsymbol{v}|^2 + U\right) - p_0 = 0 \qquad \text{on} \qquad \Sigma(t) \tag{17}$$

establishing that the pressure equals to the ullage pressure p_0 *on the free surface. The volume conservation condition* (2) *should be added to the sloshing problem.*

In summary, utilizing the Clebsch potentials and the Bateman–Luke principle (*the Lagrangian is a "pressure integral"*) makes it possible to derive the full system of governing equations (8), (11), (13) and the boundary conditions (9) and (17) for the sloshing of an ideal incompressible liquid with *rotational flows*. Specifically, the principle (integrand in Lagrangian (5) is the pressure) holds true if and only if the vorticity equations (11) and (13) are *a priori* satisfied. The generalized Bateman–Luke formulation can be a background for the nonlinear multimodal method.

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Формалізм Бейтмена–Люка в задачах хлюпання рідини з вихоровими течіями

Базуючись на представленні поля швидкостей через потенціали Клебша, узагальнюється варіаційний формалізм Бейтмена–Люка для задачі про хлюпання ідеальної нестисливої рідини з вихоровими течіями.

Ключові слова: варіаційний принцип Бейтмена–Люка, хлюпання рідини, потенціали Клебша.

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Формализм Бейтмена–Люка в задачах плескания жидкости с вихревыми течениями

Базируясь на представлении поля скоростей через потенциалы Клебша, обобщается вариационный формализм Бейтмена–Люка для задачи о плескании идеальной несжимаемой жидкости с вихревыми течениями.

Ключевые слова: вариационный принцип Бейтмена–Люка, плескание жидкости, потенциалы Клебша.