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L.A. Kurdachenko [1], I.Ya. Subbotin [2], V.S. Yashchuk [1]

[1] Oles Honchar Dnipro National University
[2] National University, Los Angeles, USA
E-mail: lkurdachenko@i.ua, isubboti@nu.edu, ViktoriiaYashchuk@mail.ua

On Leibniz algebras, whose subideals are ideals

Presented by Corresponding Member of the NAS of Ukraine V.P. Motornyi

We obtain a description of solvable Leibniz algebras, whose subideals are ideals. A description of certain types of Leibniz T-algebras is also obtained. In particular, it is established that the structure of Leibniz T-algebras essentially depends on the structure of its nil-radical.

Keywords: *Leibniz algebra, ideal, subideal, T-algebra.*

An algebra L over a field F is said to be a *Leibniz algebra* (more precisely, a *left Leibniz algebra*) if it satisfies the Leibniz identity

$$[[a, b], c] = [a, [b, c]] - [b, [a, c]] \text{ for all } a, b, c \in L. \qquad \text{(LI)}$$

Leibniz algebras are generalizations of Lie algebras. Indeed, a Leibniz algebra L is a Lie algebra if and only if $[a, a] = 0$ for every element $a \in L$. For this reason, we may consider Leibniz algebras as ”non-anticommutative” analogs of Lie algebras.

Leibniz algebra appeared first in the papers by A.M. Bloh ([1—3]), in which he called them D-algebras. However, in that time these works were not in demand, and they have not been properly developed. Real interest in Leibniz algebras arose only after two decades thanks to the work by J.L. Loday [4], who “rediscovered” these algebras and used the term *Leibniz algebras,* since it was Leibniz who discovered and proved the ”Leibniz rule” for the differentiation of functions. The Leibniz algebras appeared to be naturally related to several topics such as differential geometry, homological algebra, classical algebraic topology, algebraic K-theory, loop spaces, noncommutative geometry, and so on. They found some applications in physics (see, e. g., ([5—7]). The theory of Leibniz algebras has been developing quite intensively but very uneven. However, there are problems natural for any algebraic structure that were not previously considered for Leibniz algebras. They are the topics concerned with the relationship of subalgebras, ideals, and subideals. It should be noted, for example, that a natural question about the structure of Leibniz algebras, whose subalgebras are ideals, was only recently considered in work [8]. If, in the case of Lie algebras, the structure of similar algebras is very simple (they are Abelian), the picture in the Leibniz algebras is more sophisticated and interesting. We have an analogous situation for cyclic subalgebras: if, in Lie algebras, every cyclic subalgebra has dimension 1, in Leibniz algebras, a cyclic subalgebra can be of very complicated structure [9].

Let L be a Leibniz algebra over a field F. If A, B are subspaces of L, then $[A, B]$ will denote a subspace generated by all elements $[a, b]$, where $a \in A$, $b \in B$. As usual, a subspace A of L is called a *subalgebra* of L if $[x, y] \in A$ for every $x, y \in A$. It follows that $[A, A] \leqslant A$.

Let M be a non-empty subset of L. Then $\langle M \rangle$ denotes the subalgebra of L generated by M.

A subalgebra A is called a *left* (respectively, *right*) ideal of L if $[y, x] \in A$ (respectively, $[x, y] \in A$) for every $x \in A$, $y \in L$. In other words, if A is a left (respectively right) ideal, then $[L, A] \leqslant A$ (respectively $[A, L] \leqslant A$).

A subalgebra A of L is called an *ideal* of L (more precisely, *two-sided ideal*) if it is both a left ideal and a right ideal, i. e., $[x, y], [y, x] \in A$ for every $x \in A$, $y \in L$.

If A is an ideal of L, we can consider a *factor-algebra* L/A. It is not hard to see that this factor-algebra also is a Leibniz algebra.

Note that the relation "to be a subalgebra of a Leibniz algebra" is transitive. However, for Lie algebras, the relation "to be an ideal" is not transitive.

Therefore, it is natural to ask the question about the structure of Leibniz algebras, in which the relation "to be an ideal" is transitive.

In this context, the following important type of subalgebras naturally arises. A subalgebra A is called a *left* (respectively, *right*) *subideal* of L if there is a finite series of subalgebras

$$A = A_0 \leqslant A_1 \leqslant \ldots \leqslant A_n = L$$

such that A_{j-1} is a left (respectively, right) ideal of A_j, $1 \leqslant j \leqslant n$.

Similarly, a subalgebra A is called a *subideal* of L if there is a finite series of subalgebras

$$A = A_0 \leqslant A_1 \leqslant \ldots \leqslant A_n = L$$

such that A_{j-1} is an ideal of A_j, $1 \leqslant j \leqslant n$.

The natural question on Leibniz algebras, in which the relation "to be an ideal" is transitive, arises.

A Leibniz algebra L is called a *T-algebra* if a relation "to be an ideal" is transitive. In other words, if A is an ideal of L and B is an ideal of A, then B is an ideal of L. It follows that, in a Leibniz T-algebra, every subideal is an ideal.

Lie T-algebras have been studied by I. Stewart [10] and A.G. Gein and Yu.N. Muhin [11]. In particular, soluble T-algebras and finite-dimensional T-algebras over a field of characteristic 0 were described.

In this paper, we will consider some generalized soluble Leibniz T-algebras. Let L be a Leibniz algebra. We define the lower central series of L

$$L = \gamma_1(L) \geqslant \gamma_2(L) \geqslant \ldots \geqslant \gamma_\alpha(L) \geqslant \gamma_{\alpha+1}(L) \geqslant \ldots \geqslant \gamma_\delta(L)$$

by the following rule: $\gamma_1(L) = L$, $\gamma_2(L) = [L, L]$, and, recursively, $\gamma_{\alpha+1}(L) = [L, \gamma_\alpha(L)]$ for all ordinals α, $\gamma_\lambda(L) = \cap_{\mu < \lambda} \gamma_\mu(L)$ for the limit ordinals λ. It is possible to show that every term of this series is an ideal of L. The last term $\gamma_\delta(L)$ is called the *lower hypocenter* of L. We have $\gamma_\delta(L) = [L, \gamma_\delta(L)]$.

If $\alpha = k$ is a positive integer, then $\gamma_k(L) = [L, [L, [L, \dots , L] \dots]$ is a left normed product of k copies of L.

A Leibniz algebra L is called *nilpotent* if there exists a positive integer k such that $\gamma_k(L) = \langle 0\rangle$. More precisely, L is said to be a *nilpotent of the nilpotency class* **c** if $\gamma_{\mathbf{c}+1}(L) = \langle 0\rangle$, but $\gamma_{\mathbf{c}}(L) \neq \langle 0\rangle$. We denote, by ncl (L), the nilpotency class of L.

As usual, a Leibniz Algebra L is called *Abelian* if $[x, y] = 0$ for all elements $x, y \in L$. In an Abelian Leibniz algebra, every subspace is a subalgebra and an ideal.

The center $\zeta(L)$ of a Leibniz algebra L is defined in the following way:

$\zeta(L) = \{x \in L \mid [x, y] = 0 = [y, x]$ for each element $y \in L\}$.

Clearly, $\zeta(L)$ is an ideal of L.

Let L be a Leibniz algebra. The subalgebra $\mathrm{Nil}(L)$ generated by all nilpotent ideals of L is called the *nil-radical* of L. Clearly $\mathrm{Nil}(L)$ is an ideal of L. If $L = \mathrm{Nil}(L)$, then L is called a Leibniz *nil-algebra*. Every nilpotent Leibniz algebra is a nil-algebra, but converse is not true even for a Lie algebra. Note also that if L is a finite-dimensional nil-algebra, then L is nilpotent.

The subalgebra $\mathrm{Ba}(L)$ generated by all nilpotent subideals of L is called the *Baer radical* of L. It is possible to show that $\mathrm{Ba}(L)$ is an ideal of L and $\mathrm{Nil}(L) \leqslant \mathrm{Ba}(L)$. If $L = \mathrm{Ba}(L)$, then L is called a Leibniz *Baer algebra*. Every nil-algebra is a Baer algebra, but converse is not true even for a Lie algebra (see, e. g., [12]). Note also that if L is a finite-dimensional Baer algebra, then L is nilpotent.

As in the cases mentioned above, the situation for the Leibniz algebra is much more complex and diverse, than it was for Lie algebras. Here are a few simple examples illustrating this point. Let F be an arbitrary field, and L be a vector space over F with a basis $\{a, c\}$. Define the operation $[\ ,\ ]$ on L by the following rule: $[a, a] = c$, $[c, a] = [a, c] = [c, c] = 0$. Then L is a cyclic Leibniz algebra, and Fc is its unique non-zero subalgebra. Moreover, Fc is the center of L, in particular, Fc is an ideal of L. Thus, every subalgebra of L is an ideal.

Let now $F = F_2$ and let L be a Leibniz algebra constructed above. Put $A = L \oplus Fv$ and let $[v, v] = [v, c] = [c, v] = 0$, $[v, a] = [a, v] = a$. It is not hard to check that A is a Leibniz algebra and L is an ideal of A. Moreover, if B is a non-zero ideal of A and L does not include B, then $B = A$. As we have seen above, Fc is a unique non-zero ideal of L. But $Fc = \zeta(L)$, thus, Fc is an ideal of A. Thus, A is a Leibniz T-algebra.

Let again $F = F_2$ and $D = L \oplus Fu$. Put now $[u, u] = [u, c] = [c, u] = 0$, $[u, a] = [a, u] = a + c$. It is not hard to check that D is a Leibniz algebra and L is an ideal of A. As above, we can check that D is a Leibniz T-algebra.

As we will see further, these examples are typical in some sense.

A Leibniz algebra L is called *extraspecial* if it satisfies the following condition:

- $\zeta(L)$ is non-trivial and has dimension 1;
- $L/\zeta(L)$ is Abelian.

Theorem A. *Let L be a Leibniz T-algebra over a field F. If L is a Baer algebra, then either L is Abelian or $L = E \oplus Z$, where $Z \leqslant \zeta(L)$ and E is an extraspecial subalgebra such that $[a, a] \neq 0$ for every element $a \notin \zeta(E)$.*

A Leibniz algebra L is called *hyper-Abelian* if it has an ascending series

$$\langle 0\rangle = L_0 \leqslant L_1 \leqslant \ldots \leqslant L_\alpha \leqslant L_{\alpha+1} \leqslant \ldots \leqslant L_\gamma = L$$

of ideals, whose factors $L_{\alpha+1}/L_\alpha$ are Abelian for all $\alpha < \gamma$. If this series is finite, then L is called a *soluble Leibniz algebra*.

The structure of Leibniz T-algebras essentially depends of the structure of its nil-radical.

Theorem B. *Let L be a hyper-Abelian Leibniz T-algebra over a field F. If L is non-nilpotent and* $\mathrm{Nil}(L) = D$ *is Abelian, then* $L = D \oplus V$, *where* $V = Fv$, $[v, v] = 0$, $[v, d] = d = -[d, v]$ *for every element* $d \in \mathrm{Nil}(L)$. *In particular, L is a Lie algebra.*

Theorem C. *Let L be a hyper-Abelian Leibniz T-algebra over a field F. If* $\mathrm{char}(F) \neq 2$, *then* $\mathrm{Nil}(L)$ *is Abelian.*

We say that a field F is *2-closed*, if the equation $x^2 = a$ has a solution in F for every element $a \neq 0$. We note that every locally finite (in particular, finite) field of characteristic 2 is 2-closed.

Theorem D. *Let L be a hyper-Abelian Leibniz T-algebra over a field F. Suppose that L is non-nilpotent and* $\mathrm{Nil}(L)$ *is non-Abelian. If the field F is* 2*-closed and* $\mathrm{char}(F) = 2$, *then* $L = (Fe \oplus Fc) \oplus Fv$, *where*

$$[e, e] = c,\ [c, e] = [e, c] = [c, v] = [v, c] = 0,$$
$$[v, v] = 0,\ [v, e] = e + \gamma c = [e, v],\ \gamma \in F.$$

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Л.А. Курдаченко [1], *І.Я. Субботін* [2], *В.С. Ящук* [1]

[1] Дніпровський національний університет ім. Олеся Гончара
[2] Національний університет, Лос-Анджелес, США
E-mail: lkurdachenko@i.ua, isubboti@nu.edu, ViktoriiaYashchuk@mail.ua

ПРО АЛГЕБРИ ЛЕЙБНІЦА,
КОЖЕН ПІДІДЕАЛ ЯКИХ Є ІДЕАЛОМ

Отримано опис розв'язних алгебр Лейбніца, всі підідеали яких є ідеалами. Наведено теореми, що дають опис деяких типів *T*-алгебр Лейбніца. Зокрема, структура *T*-алгебр Лейбніца істотно залежить від структури її ніль-радикала.

Ключові слова: *алгебра Лейбніца, ідеал, підідеал, T-алгебра.*

Л.А. Курдаченко [1], *И.Я. Субботин* [2], *В.С. Ящук* [1]

[1] Днепровский национальный университет им. Олеся Гончара
[2] Национальный университет, Лос-Анджелес, США
E-mail: lkurdachenko@i.ua, isubboti@nu.edu, ViktoriiaYashchuk@mail.ua

ОБ АЛГЕБРАХ ЛЕЙБНИЦА,
КАЖДЫЙ ПОДИДЕАЛ КОТОРЫХ ЯВЛЯЕТСЯ ИДЕАЛОМ

Получено описание разрешимых алгебр Лейбница, все подидеалы которых являются идеалами. Приведены теоремы, которые дают описание некоторых типов *T*-алгебр Лейбница. В частности, структура *T*-алгебр Лейбница существенно зависит от структуры ее ниль-радикала.

Ключевые слова: *алгебра Лейбница, идеал, подидеал, T-алгебра.*