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V.V. Bilet, O.A. Dovgoshey

Institute of Applied Mathematics and Mechanics of the NAS of Ukraine, Slovyansk
E-mail: viktoriiabilet@gmail.com, oleksiy.dovgoshey@gmail.com

Asymptotic behavior of metric spaces at infinity

Presented by Corresponding Member of the NAS of Ukraine V. Ya. Gutlyanskii

A new sequential approach to investigations of the structure of metric spaces at infinity is proposed. Criteria for the finiteness and boundedness of metric spaces at infinity are found.

Keywords: *asymptotic boundedness of a metric space, asymptotic finiteness of a metric space, convergence of metric spaces, strong porosity at a point.*

As the asymptotic structure describing the behavior of an unbounded metric space (X, d) at infinity, we mean a metric space that is a limit of rescaling metric spaces $\left(X, \frac{1}{r_n} d\right)$ for r_n tending to infinity. The Gromov—Hausdorff convergence and the asymptotic cones are most often used for the construction of such limits. Both of these approaches are based on higher-order logic abstractions (see, e. g., [1] for details), which makes them very powerful, but it does away the constructiveness. In this paper, we propose a more elementary, sequential approach for describing the structure of unbounded metric spaces at infinity.

Let (X, d) be an unbounded metric space, p be a point of X, and $\tilde{r} = (r_n)_{n\in\mathbb{N}}$ be a scaling sequence of positive real numbers tending to infinity. Denote, by $\tilde{\mathbf{X}}_{\infty,\tilde{r}}$, the set of all sequences $\tilde{x} = (x_n)_{n\in\mathbb{N}} \subset X$, for each of which $\lim\limits_{n\to\infty} d(x_n, p) = \infty$, and there is a finite limit $\tilde{\tilde{d}}_{\tilde{r}}(\tilde{x}) := \lim\limits_{n\to\infty} \frac{d(x_n, p)}{r_n}$. Define the equivalence relation $\equiv$ on $\tilde{\mathbf{X}}_{\infty,\tilde{r}}$ as

$$(\tilde{x} \equiv \tilde{y}) \Leftrightarrow \left(\lim_{n\to\infty} \frac{d(x_n, y_n)}{r_n} = 0\right).$$

Let $\mathbf{\Omega}^X_{\infty,\tilde{r}}$ be the set of equivalence classes generated by $\equiv$ on $\tilde{\mathbf{X}}_{\infty,\tilde{r}}$. We say that points $\alpha, \beta \in \mathbf{\Omega}^X_{\infty,\tilde{r}}$ are *mutually stable* if, for $\tilde{x} \in \alpha$ and $\tilde{y} \in \beta$, there is a limit

$$\rho(\alpha, \beta) := \lim_{n\to\infty} \frac{d(x_n, y_n)}{r_n}. \qquad (1)$$

Let us consider the weighted graph $(G_{X,\tilde r},\rho)$ with the vertex set $V(G_{X,\tilde r})=\mathbf{\Omega}^X_{\infty,\tilde r}$, and the edge set $E(G_{X,\tilde r})$ such that

$$(\{u,v\}\in E(G_{X,\tilde r}))\Leftrightarrow(u \text{ and } v \text{ are mutually stable and } u\neq v),$$

and the weight $\rho: E(G_{X,\tilde r})\to\mathbb{R}^+$ defined by formula (1).

Definition 1. The *pretangent spaces* (to (X,d) at infinity w.r.t. $\tilde r$) are the maximal cliques $\Omega^X_{\infty,\tilde r}$ of $G_{X,\tilde r}$ with metrics determined with the help of (1).

Recall that a *clique* in a graph G is a set $A\subseteq V(G)$ such that every two distinct points of A are adjacent. A clique C in G is maximal if $C\subseteq A$ implies $C=A$ for every clique A in G.

Define a subset $\alpha_0=\alpha_0(X,\tilde r)$ of the set $\tilde{\mathbf{X}}_{\infty,\tilde r}$ by the rule

$$(\tilde z\in\alpha_0)\Leftrightarrow(\tilde z\in\tilde{\mathbf{X}}_{\infty,\tilde r} \text{ and } \tilde{\tilde d}_{\tilde r}(\tilde z)=0). \qquad (2)$$

Then α_0 is a common point of all pretangent spaces $\Omega^X_{\infty,\tilde r}$. This means, in particular, that the graph $G_{X,\tilde r}$ is connected.

Let $(n_k)_{k\in\mathbb{N}}\subset\mathbb{N}$ be infinite and strictly increasing. Denote, by $\tilde r'$, the subsequence $(r_{n_k})_{k\in\mathbb{N}}$ of $\tilde r=(r_n)_{n\in\mathbb{N}}$ and, for every $\tilde x=(x_n)_{n\in\mathbb{N}}\in\tilde{\mathbf{X}}_{\infty,\tilde r}$, write $\tilde x':=(x_{n_k})_{k\in\mathbb{N}}$. It is clear that $\lim\limits_{k\to\infty}d(x_{n_k},p)=$ $=\infty$ and $\tilde{\tilde d}_{\tilde r'}(\tilde x')=\tilde{\tilde d}_{\tilde r}(\tilde x)$ for every $\tilde x\in\tilde{\mathbf{X}}_{\infty,\tilde r}$. Moreover, if $\tilde y\in\tilde{\mathbf{X}}_{\infty,\tilde r}$ and if $\lim\limits_{n\to\infty}\dfrac{d(x_n,y_n)}{r_n}$ exists, then

$$\lim_{k\to\infty}\frac{d(x_{n_k},y_{n_k})}{r_{n_k}}=\lim_{n\to\infty}\frac{d(x_n,y_n)}{r_n}. \qquad (3)$$

Let us define $\tilde{\mathbf{X}}_{\infty,\tilde r'}$ and ρ' similarly to $\tilde{\mathbf{X}}_{\infty,\tilde r}$ and, respectively, to ρ, and let $\pi:\tilde{\mathbf{X}}_{\infty,\tilde r}\to\mathbf{\Omega}_{\infty,\tilde r}$, $\pi':\tilde{\mathbf{X}}_{\infty,\tilde r'}\to\mathbf{\Omega}_{\infty,\tilde r'}$ be the natural projections $\pi(\tilde x):=\{\tilde y\in\tilde{\mathbf{X}}_{\infty,\tilde r}:\rho(\tilde x,\tilde y)=0\}$, $\pi'(\tilde x):=\{\tilde y\in\tilde{\mathbf{X}}_{\infty,\tilde r'}:$ $:\rho'(\tilde x,\tilde y)=0\}$, and let $\varphi_{\tilde r'}(\tilde x):=\tilde x'$ for all $\tilde x\in\tilde{\mathbf{X}}_{\infty,\tilde r}$. Then there is an embedding $\mathrm{em}':\mathbf{\Omega}^X_{\infty,\tilde r}\to$ $\to\mathbf{\Omega}^X_{\infty,\tilde r'}$ of the weighted graph $(G_{X,\tilde r},\rho)$ in the weighted graph $(G_{X,\tilde r'},\rho')$ such that the diagram

$$\begin{array}{ccc}
\tilde{\mathbf{X}}_{\infty,\tilde r} & \xrightarrow{\varphi_{\tilde r'}} & \tilde{\mathbf{X}}_{\infty,\tilde r'} \\
\Big\downarrow{\scriptstyle\pi} & & \Big\downarrow{\scriptstyle\pi'} \\
\mathbf{\Omega}^X_{\infty,\tilde r} & \xrightarrow{\mathrm{em}'} & \mathbf{\Omega}^X_{\infty,\tilde r'}
\end{array}$$

is commutative. Since em' is an embedding of weighted graphs, $\mathrm{em}'(C)$ is a clique in $G_{X,\tilde r'}$ if C is a clique in $G_{X,\tilde r}$. Furthermore, (3) implies that the restrictions $\mathrm{em}'|_{\Omega^X_{\infty,\tilde r}}$ are isometries of the pretangent spaces $\mathbf{\Omega}^X_{\infty,\tilde r}$ on the metric spaces $\mathrm{em}'(\mathbf{\Omega}^X_{\infty,\tilde r})$.

Definition 2. A pretangent space $\Omega^X_{\infty,\tilde r}$ is *tangent* if the clique $\mathrm{em}'(\Omega^X_{\infty,\tilde r})$ is maximal for every infinite, strictly increasing sequence $(n_k)_{k\in\mathbb{N}}\subset\mathbb{N}$.

Example 1. Let E be a finite-dimensional Euclidean space and let $X\subseteq E$ be such that the Hausdorff distance $d_H(E,X)$ is finite. Then, for every scaling sequence $\tilde r$, all pretangent spaces $\Omega^X_{\infty,\tilde r}$ are tangent and isometric to E.

In conclusion of this brief introduction, it should be noted that there exist other techniques, which allow one to investigate the asymptotic properties of metric spaces at infinity. As examples, we mention only the balleans theory [2] and the Wijsman convergence [3—5].

1. Finiteness. In this section, we study the conditions, under which pretangent spaces are finite.

Theorem 1. *Let* (X, d) *be an unbounded metric space,* $p \in X$, $n \geqslant 2$, *and let*

$$F_n(x_1, \ldots, x_n) := \begin{cases} \dfrac{\min\limits_{1 \leqslant k \leqslant n} d(x_k, p) \prod\limits_{1 \leqslant k < l \leqslant n} d(x_k, x_l)}{\left(\max\limits_{1 \leqslant k \leqslant n} d(x_k, p)\right)^{\frac{n(n-1)}{2}+1}}, & \text{if } (x_1, \ldots, x_n) \neq (p, \ldots, p), \\ 0, & \text{if } (x_1, \ldots, x_n) = (p, \ldots, p). \end{cases}$$

Then the inequality $\left|\Omega^X_{\infty, \tilde{r}}\right| \leqslant n$ *holds for every pretangent space* $\Omega^X_{\infty, \tilde{r}}$ *if and only if* $\lim\limits_{x_1, \ldots, x_n \to \infty} F_n(x_1, \ldots, x_n) = 0.$

Note that, for every unbounded metric space (X, d), there is a pretangent space $\Omega^X_{\infty, \tilde{r}}$ consisting of at least two points.

Corollary 1. *Let* (X, d) *be an unbounded metric space and let* α_0 *be a point defined by* (2) *for every scaling sequence* $\tilde{r}$. *Then the following statements are equivalent:*

(i) *The graph* $G_{X, \tilde{r}}$ *is a star with the center* α_0 *for every scaling sequence* $\tilde{r}$*;*

(ii) *The limit relation* $\lim\limits_{x_1, x_2 \to \infty} F_2(x_1, x_2) = 0$ *holds.*

Let us consider now the problem of existence of finite *tangent* spaces.

Definition 3. Let $E \subseteq \mathbb{R}^+$. The *porosity* of E at infinity is the quantity

$$p(E, \infty) := \limsup_{h \to \infty} \frac{l(\infty, h, E)}{h}, \tag{4}$$

where $l(\infty, h, E)$ is the length of the longest interval in the set $[0, h] \setminus E$. The set E is *strongly porous* at infinity if $p(E, \infty) = 1$.

The standard definition of the porosity at a point can be found in [6].

For a metric space (X, d) and $p \in X$, we write $S_p(X) := \{d(x, p) : x \in X\}$.

Theorem 2. *Let* (X, d) *be an unbounded metric space,* $p \in X$. *The following statements are equivalent:*

(i) *The set* $S_p(X)$ *is strongly porous at infinity;*

(ii) *There is a single-point tangent space* $\Omega^X_{\infty, \tilde{r}}$*;*

(iii) *There is a finite tangent space* $\Omega^X_{\infty, \tilde{r}}$*;*

(iv) *There is a compact tangent space* $\Omega^X_{\infty, \tilde{r}}$*;*

(v) *There is a bounded, separable tangent space* $\Omega^X_{\infty, \tilde{r}}$.

Some results similar to Theorem 1 and Theorem 2 can be found in [7] and [8] respectively.

2. Boundedness. Let $\tilde{\tau} = (\tau_n)_{n \in \mathbb{N}} \subset \mathbb{R}$. We say that $\tilde{\tau}$ is eventually increasing if the inequality $\tau_{n+1} \geqslant \tau_n$ holds for sufficiently large n. For $E \subseteq \mathbb{R}^+$, we write $\tilde{E}^i_\infty$ for the set of eventually increasing sequences $\tilde{\tau} \subset E$ with $\lim\limits_{n\to\infty} \tau_n = \infty$. Denote also, by $\tilde{I}_E^i$, the set of all sequences of open intervals $(a_n, b_n) \subseteq \mathbb{R}^+$ meeting the following conditions:

Each (a_n, b_n) is a connected component of the set $\mathrm{Int}(\mathbb{R}^+ \setminus E)$;

$(a_n)_{n\in\mathbb{N}}$ is eventually increasing;

$$\lim_{n\to\infty} a_n = \infty \text{ and } \lim_{n\to\infty} \frac{b_n - a_n}{b_n} = 1.$$

Define an equivalence $\asymp$ on the set of sequences of strictly positive numbers as follows. Let $\tilde{a} = (a_n)_{n\in\mathbb{N}}$ and $\tilde{\gamma} = (\gamma_n)_{n\in\mathbb{N}}$. Then $\tilde{a} \asymp \tilde{\gamma}$ if there are some constants $c_1, c_2 > 0$ such that $c_1 a_n < \gamma_n < c_2 a_n$ for every $n \in \mathbb{N}$.

Definition 4. Let $E \subseteq \mathbb{R}^+$ and let $\tilde{\tau} \in \tilde{E}_\infty^i$. The set E is $\tilde{\tau}$*-strongly porous* at infinity if there is a sequence $((a_n, b_n))_{n\in\mathbb{N}} \in \tilde{I}_E^i$ such that $\tilde{\tau} \asymp \tilde{a}$, where $\tilde{a} = (a_n)_{n\in\mathbb{N}}$. The set E is *completely strongly porous* at infinity if E is $\tilde{\tau}$-strongly porous at infinity for every $\tilde{\tau} \in \tilde{E}_\infty^i$.

Note that every set completely strongly porous at infinity is strongly porous at infinity, but not conversely.

Definition 5. Let (X, d) be an unbounded metric space and let $p \in X$. A scaling sequence $\tilde{r}$ is *normal* if $\tilde{r}$ is eventually increasing and there is $\tilde{x} \in \tilde{\mathbf{X}}_{\infty, \tilde{r}}$ such that

$$\lim_{n\to\infty} \frac{d(x_n, p)}{r_n} = 1.$$

Write $\mathfrak{I}_n(X)$ for the set of all pretangent spaces $\Omega_{\infty, \tilde{r}}^X$ with normal scaling sequences $\tilde{r}$. Under which conditions is the family $\mathfrak{I}_n(X)$ uniformly bounded?

Recall that a family $\mathfrak{I}$ of a metric spaces (Y, d_Y) is *uniformly bounded* if

$$\sup\{\operatorname{diam} Y : Y \in \mathfrak{I}\} < \infty$$

If all metric spaces (Y, d_Y) are pointed with marked points $p_Y \in Y$ and

$$\inf\{d_Y(p_Y, y) : y \in Y \setminus \{p_Y\}, Y \in \mathfrak{I}\} > 0,$$

then we say that $\mathfrak{I}$ is *uniformly discrete* (w.r.t. the points p_Y).

The following theorem is an analog of Theorem 3.11 and Theorem 4.1 from [9].

Theorem 3. *Let (X, d) be an unbounded metric space and let $p \in X$ Then the following statements are equivalent:*

(i) *The family $\mathfrak{I}_n(X)$ is uniformly bounded;*

(ii) *$S_p(X)$ is completely strongly porous at infinity;*

(iii) *The family $\mathfrak{I}_n(X)$ is uniformly discrete w.r.t. the points α_0 defined by (2).*

If $\mathfrak{I}_n(X)$ is uniformly bounded, then every pretangent space $\Omega_{\infty, \tilde{r}}^X$ is bounded, but the converse, in general, does not hold.

Definition 6. The set $E \subseteq \mathbb{R}^+$ is w*-strongly porous* at infinity if, for every sequence $\tilde{\tau} \in \tilde{E}_\infty^i$, there is a subsequence $\tilde{\tau}'$, for which E is $\tilde{\tau}'$-strongly porous at infinity.

The following theorem gives a boundedness criterion for pretangent spaces.

Theorem 4. *Let (X, d) be an unbounded metric space and let $p \in X$. All pretangent spaces to X at infinity are bounded if and only if the set $S_p(X)$ is w-strongly porous at infinity.*

The example of w-strongly porous at infinity set $E \subseteq \mathbb{R}^+$, which is not completely strongly porous at infinity, can be obtained as a modification of Example 2.10 [10].

Using Theorem 4, we can obtain a criterion of existence of an unbounded pretangent space. The conditions of such type are important for the development of a theory of pretangent spaces of the second order or more (i.e., pretangent spaces to pretangent spaces, pretangent spaces to pretangent spaces to pretangent spaces, and so on).

Let (Y, d_Y) be a metric space. Then, for every $K \subseteq Y$ and $y \in Y$, we write $\operatorname{dist}(y, K) = \inf\{d_Y(y, x) : x \in K\}$. The following definition can be found in [11].

Definition 7. *Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of subsets of (Y, d_Y) The set*

$$\operatorname*{Lim\,inf}_{n\to\infty} K_n := \{y \in Y : \lim_{n\to\infty} \operatorname{dist}(y, K_n) = 0\}$$

is the *Kuratowski lower limit* of $(K_n)_{n\in\mathbb{N}}$ in (Y, d_Y).

For $A \subseteq \mathbb{R}$ and $t \in \mathbb{R}$, we set $tA := \{ta : a \in A\}$.

Theorem 5. *Let (X, d) be an unbounded metric space and let $p \in X$. Then the following statements are equivalent:*

(i) *There exists an unbounded pretangent space $\Omega^X_{\infty,\tilde{r}}$;*

(ii) *There exists a scaling sequence $\tilde{r} = (r_n)_{n\in\mathbb{N}}$ such that the Kuratowski lower limit $\operatorname*{Lim\,inf}_{n\to\infty} \frac{1}{r_n} S_p(X)$ is an unbounded subset of $\mathbb{R}^+$;*

(iii) *The set $S_p(X)$ is not w-strongly porous at infinity.*

Let $(\tilde{r}_m)_{m\in\mathbb{N}}$ be a sequence of scaling sequences. Then, for every $m \in \mathbb{N}$ and every unbounded metric space (X, d), we define a pretangent space $\Omega^X_{\infty,(\tilde{r}_1,\ldots,\tilde{r}_m)}$ by the following inductive rule: $\Omega^X_{\infty,(\tilde{r}_1)} := \Omega^X_{\infty,\tilde{r}_1}$ if $m = 1$ and

$$\Omega^X_{\infty,(\tilde{r}_1,\ldots,\tilde{r}_m)} := \Omega^{\Omega^X_{\infty,(r_1,\ldots,r_{m-1})}}_{\infty,\tilde{r}_m}$$

if $m \geqslant 2$.

Using Theorem 5, we can obtain the following corollary.

Corollary 2. *Let (X, d) be an unbounded metric space and let $p \in X$. If the equality $p(S_p(X), \infty) = 0$ holds, then there is a sequence $(\tilde{r}_m)_{m\in\mathbb{N}}$ of scaling sequences such that the pretangent space $\Omega^X_{\infty,(\tilde{r}_1,\ldots,\tilde{r}_m)}$ is unbounded for every $m \in \mathbb{N}$.*

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В.В. Білет, О.А. Довгоший

Інститут прикладної математики і механіки НАН України, Слов'янськ
E-mail: viktoriiabilet@gmail.com, oleksiy.dovgoshey@gmail.com

АСИМПТОТИЧНА ПОВЕДІНКА
МЕТРИЧНИХ ПРОСТОРІВ НА НЕСКІНЧЕННОСТІ

Запропоновано новий секвенціальний підхід до дослідження структури метричних просторів у нескінченно віддаленій точці. Знайдено критерії скінченності та обмеженості метричних просторів на нескінченності.

Ключові слова: *асимптотична обмеженість метричного простору, асимптотична скінченність метричного простору, збіжність метричних просторів, сильна пористість у точці.*

В.В. Билет, А.А. Довгошей

Институт прикладной математики и механики НАН Украины, Славянск
E-mail: viktoriiabilet@gmail.com, oleksiy.dovgoshey@gmail.com

АСИМПТОТИЧЕСКОЕ ПОВЕДЕНИЕ
МЕТРИЧЕСКИХ ПРОСТРАНСТВ НА БЕСКОНЕЧНОСТИ

Предложен новый секвенциальный подход к исследованию структуры метрических пространств в бесконечно удаленной точке. Найдены критерии конечности и ограниченности метрических пространств на бесконечности.

Ключевые слова: *асимптотическая ограниченность метрического пространства, асимптотическая конечность метрического пространства, сходимость метрических пространств, сильная пористость в точке.*