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V.P. Sylovanyuk, https://orcid.org/0000-0002-3765-5962
A.I. Didukh, https://orcid.org/0009-0005-2265-7411
Karpenko Physico-Mechanical Institute of the NAS of Ukraine, Lviv, Ukraine
E-mail: dandriy066@gmail.com

Healing of penny-shaped crack in material with functional gradient structure

Presented by Corresponding Member of the NAS of Ukraine O.Ye. Andreikiv

Composites with a continuously varying volume fraction are called functionally graded materials. As part of research on functionally graded materials, the influence of a continuous axial gradient in elastic properties and a thin, ductile repair layer on a penny-shaped crack is being examined. The body is modeled as an isotropic solid whose shear modulus varies along an axis perpendicular to the crack according to a power-law relationship, while Poisson's ratio is fixed. The solidified injected filler occupies the crack cavity as a thin layer of variable thickness and transmits tensile forces between the crack surfaces in accordance with the Winkler-type contact reaction. Hankel transforms applied together with the interlayer-modified condition on the crack faces, reduce the problem to a Fredholm integral equation of the second kind for the crack-opening displacement. The equation is solved numerically by a quadrature method, and the normalized stress intensity factor is evaluated as a function of the grading parameter, the filler stiffness and the relative interlayer thickness. The analysis shows that the solution depends on the filler only through an effective stiffness combining these two parameters, so filler stiffness and interlayer thinness are interchangeable: the thinner the interlayer, the more compliant the filler that suffices for the same healing effect. Numerical results show that the recovery effect is sharply nonlinear: even a very ductile filler eliminates most of the stress concentration near the crack perimeter, whereas a further increase in the filler's stiffness or a reduction in the interlayer thickness yields an increasingly smaller effect. For very thin interlayers, the residual stress concentration decreases to a small fraction of the level characteristic of an open crack. Material grading increases the stress intensity level in an unhealed crack; however, the relative effectiveness of healing increases as the degree of grading increases: when using the same filler, the residual SIF in a body with a strong gradation decreases to a smaller fraction of its value in the ungraded state than in a homogeneous body; therefore, grouting by injection proves to be even more effective in functionally graded materials. The results obtained confirm the possibility of strengthening, through injection, cracked structural elements made of continuously heterogeneous materials.

Keywords: *crack, functionally graded material, crack healing, stress intensity factor.*

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Introduction. To meet the high demands of modern and, in particular, future technologies in the energy, transportation, construction, and aerospace sectors, it is necessary to develop new materials with a functionally graded structure tailored to specific applications. Such materials are primarily composites or intermetallics with a smoothly varying volume fraction of one or more phases. A number of problems in the field of fracture mechanics for these materials still require investigation. These include identifying the interaction between continuous material heterogeneity and cracks, including prolonged cracks, as well as determining the characteristics of their deformation and fracture near defects.

The present work investigates the extent to which the load-bearing capacity of a continuously heterogeneous fractured body can be restored by injecting a restorative material into the fracture cavity. For homogeneous materials, similar studies have been conducted in a number of papers [1—5] and have demonstrated the high effectiveness of injection-based reinforcement technology for crack-damaged isotropic and anisotropic materials.

Problem formulation and methods of research. Let a long cylinder of radius R contain a penny-shaped crack of radius a, as shown in Fig. 1, located in the plane $z = 0$ of the Cartesian coordinate system (x, y, z).

The material of the cylinder is isotropic and nonhomogeneous, with the shear modulus varying continuously along the z-axis according to the power law $\mu(z) = \mu_0|z/a|^n$, while Poisson's ratio ν_0 remains constant; μ_0 is a constant; $0 \le n \le 1$. Tensile forces of intensity p are applied to the ends of the cylinder farthest from the crack. No external forces act on the lateral surfaces of the cylinder.

For the case when the crack size is small compared with the diameter of the cylinder ($R >> a$), an expression for the asymptotic representation of the normal stresses σ_z near the crack contour was obtained in [6]:

$$\sigma_z(r,0) = \frac{k_1}{(2r_1)^{\beta}} + O(r_1^0), \quad \beta = 0.5(1-n), \quad 0 \le \beta \le 0.5, \tag{1}$$

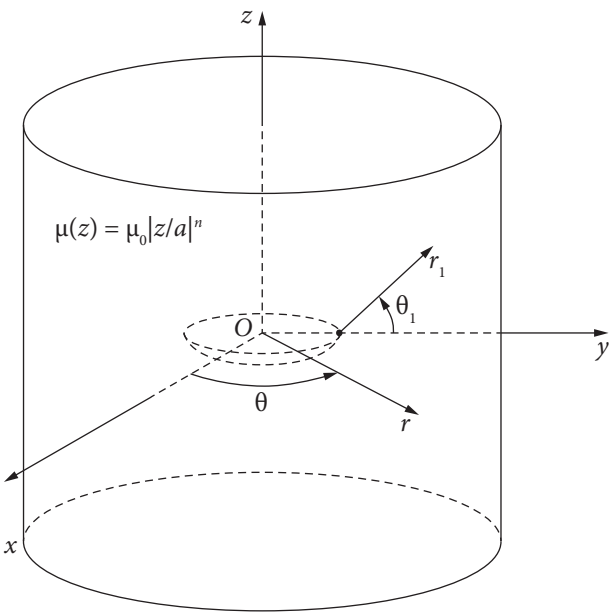


Fig. 1. Geometry of a penny-shaped crack in a continuously nonhomogeneous cylindrical body under tension

where the coefficient k_1 takes the value

$$k_1 = \frac{\sin(\pi\beta)}{\pi\beta} p a^{\beta}. \tag{2}$$

Note that the singularity of the normal stresses σ_z, as follows from (1), is r_1^{β} and differs from the singularity of the stresses σ_z in a homogeneous material, which is known to be $r_1^{-1/2}$.

Let the crack cavity be filled with an injection material that, upon hardening, bonds the opposite crack faces and forms a healing interlayer of thickness $2h(r)$. Its mechanical response is described according to the Winkler model:

$$\sigma_z(r,0) = \frac{u_z(r) E_1}{h(r)};$$

E_1 is the Young's modulus of the injection material.

By virtue of the symmetry of the problem with respect to the plane $z = 0$, the problem can be reduced to the following boundary value problem for the half-space $z \geq 0$ with mixed conditions:

$$\sigma_z(r,0) = -p + \frac{u_z(r) E_1}{h(r)}, \quad r < a; \quad \tau_{rz}(r,0) = 0, \quad 0 \leq r < \infty; \quad u_z(r,0) = 0, \quad r \geq a. \tag{3}$$

Problem solution. With the assumption that

$$\mu(z) = \mu_0 |z/a|^n, \quad 0 \leq n \leq 1, \qquad \nu = \nu_0 = 1/(2+n), \tag{4}$$

equilibrium equations of the nonhomogeneous body, with axial symmetry of the problem considered, take the form:

$$\nabla^2 u_r + \frac{1}{1-2\nu_0}\frac{\partial e}{\partial r} - \frac{u_r}{r^2} + \frac{n}{z}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) = 0, \tag{5a}$$

$$\nabla^2 u_z + \frac{1}{1-2\nu_0}\frac{\partial e}{\partial z} + \frac{2n}{z}\left[\frac{\nu_0 e}{1-2\nu_0} + \frac{\partial u_z}{\partial z}\right] = 0, \quad e = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}, \tag{5b}$$

where ∇^2 is the Laplace operator in cylindrical polar coordinates $\nabla^2 = \partial^2/\partial r^2 + (1/r)\partial/\partial r + \partial^2/\partial z^2$.

Due to the symmetry of the problem with respect to the plane $z = 0$ the shear stress vanishes on the crack plane, the representations [6]:

$$u_r(r,z) = z\frac{\partial \psi}{\partial r}, \quad u_z(r,z) = -\psi + z\frac{\partial \psi}{\partial z}, \tag{6}$$

satisfy equations (5) if

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{n}{z}\frac{\partial}{\partial z} + \frac{\partial^2}{\partial z^2}\right)\psi(r,z) = 0. \tag{7}$$

It follows from (7) that the solution for ψ, has the form:

$$\psi(r,z)=z^{\beta}\int_0^{\infty}s^{\beta}D(s)K_{\beta}(sz)J_0(sr)\,\mathrm{d}s,\qquad \beta=0.5(1-n),\quad 0\le\beta\le 0.5, \tag{8}$$

in which K_β denotes the modified Bessel function of the second kind of order β, J_0 is the Bessel function of the first kind of order zero and $D(s)$ an unknown spectral density fixed by the mixed conditions (3).

Enforcement of conditions (3) on the representation (8) produces the system of dual integral equations

$$\int_0^{\infty}D(s)J_0(rs)\,\mathrm{d}s=0,\quad r\ge a, \tag{9a}$$

$$\int_0^{\infty}s^{2\beta}D(s)J_0(rs)\,\mathrm{d}s=\frac{a^n}{2^{2-\beta}\mu_0\Gamma(2-\beta)}\left[p-\frac{u_z(r)E_1}{h(r)}\right],\quad r<a, \tag{9b}$$

Γ (2 − β) — gamma function. Copson's reduction [7] then converts (9) into a Fredholm integral equation of the second kind for the crack-opening displacement

$$u_z(r)=-a^n\pi 2^{\beta-1}\Big/(\sin(\pi\beta)\Gamma(1-\beta))\int_0^{\infty}s^{1-\beta}\int_0^{a}\frac{t^{1-\beta}\sin(\pi\beta)}{2\pi\mu_0(1-\beta)}\times$$

$$\times\int_0^{t}\frac{\rho(-p+E_1u_z(\rho)/h(\rho))d\rho}{(t^2-\rho^2)^{1-\beta}}\,J_{\beta}(st)\,dtJ_0(rs)\,ds. \tag{10}$$

The stress intensity factor, determined from the asymptotics (1) with the crack opening $u_z(r)$ found from (10), takes the form

$$k_1=\frac{pa^{\beta}\sin(\pi\beta)}{\pi\beta}-\frac{2\mu_0\sin(\pi\beta)\varepsilon}{\pi a^{\beta}}\int_0^{a}\frac{ru_z(r)}{h(r)(a^2-r^2)^{1-\beta}}\,\mathrm{d}r, \tag{11}$$

where $\varepsilon=E_1/\mu_0$ is dimensionless filler stiffness.

Numerical results and discussion. Equation (10) was solved using the Nyström method (a quadrature method) with collocation points coinciding with the quadrature nodes. The composite Gauss—Legendre rule was used. The diagonal elements of the weakly singular kernel were fixed using the exact identity of the free term (the solution in closed form for ε = 0). The accuracy was verified by recomputing the solution on a refined grid by a factor of two, the difference did not exceed $1\cdot 10^{-4}$.

The hardened filler is modelled as an interlayer of constant thickness $h(r)=h_0$, characterized by the relative thinness $\lambda=a/h_0$; the solution depends on the filler only through the product ελ. The following parameter values were taken: for the stiffness sweep, an interlayer with $h_0=a/100$ (λ = 100) and ε from 0 to 0.1; for the thinness sweep, λ from 10^4 to 10^5 (ultra-thin

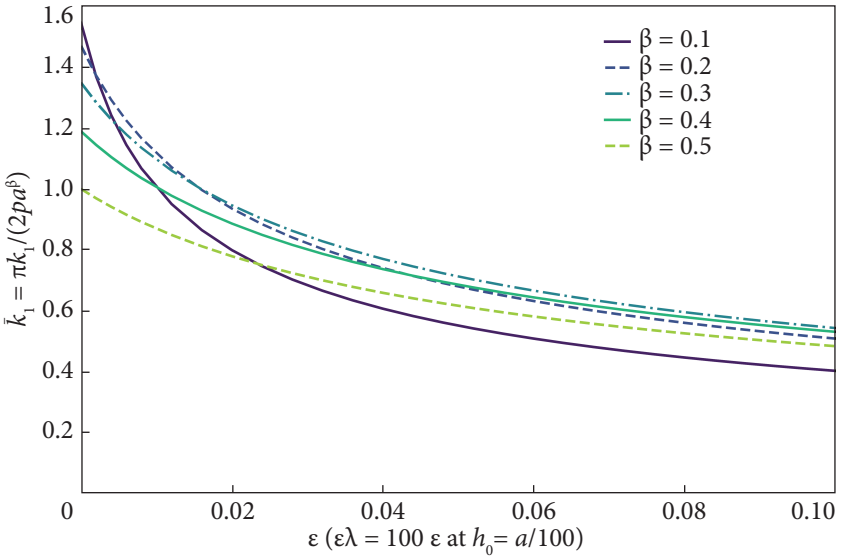


Fig. 2. Normalized SIF $\bar{k}_1$ versus the filler stiffness ε. Constant-thickness film $h = h_0$, λ = 100

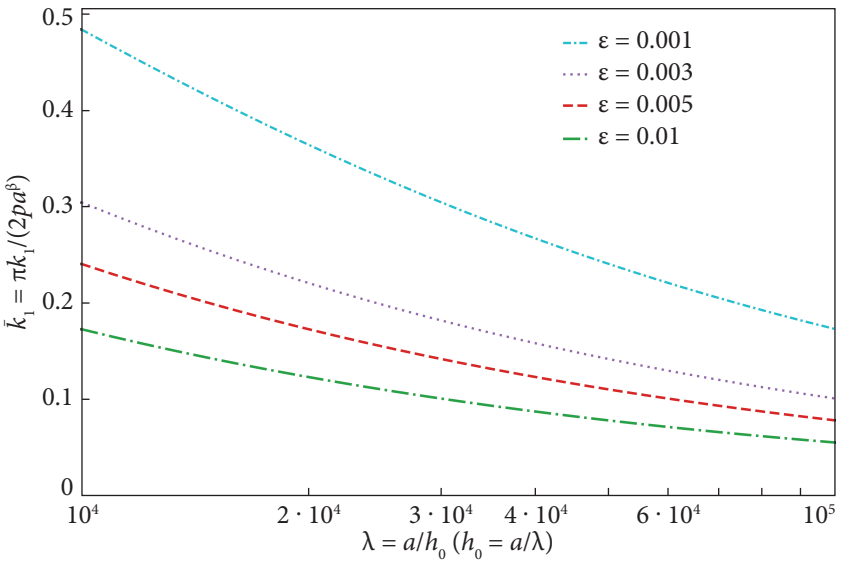


Fig. 3. Normalized SIF $\bar{k}_1$ versus the relative interlayer thinness $\lambda = a/h_0$. Constant-thickness film $h = h_0$, β = 0.5

interlayers, $h_0 = a/10^4 \ldots a/10^5$) with ε = 0.001, 0.003, 0.005 and 0.01; the nonhomogeneity exponent β from 0.1 to 0.5.

Fig. 2 shows the dependence of the normalized SIF $\bar{k}_1$ on the filler stiffness ε at λ = 100 for β = 0.1…0.5. The dependence is sharply nonlinear: the sharpest decrease occurs at the very beginning of the stiffness range, so even an extremely compliant filler eliminates most of the stress

concentration, whereas as stiffness increases further, the curves level off, and the SIF approaches zero only asymptotically.

Fig. 3 shows $\bar{k}_1$ over the whole range $\lambda = 10^4 \ldots 10^5$ at $\beta = 0.5$. The curves for different stiffnesses are nested and do not intersect: a stiffer filler retains its advantage over the whole range of thinness, while the relative benefit of thinning is the same for every stiffness, so a stiffer injection material and a thinner hardened interlayer may be traded against each other.

Conclusions. The influence of the material gradient and the stiffness of the filler on the normalized stress intensity factor was analyzed. The homogeneous body is contained as the case $\beta = 0.5$. For an unhealed crack ($\varepsilon = 0$) the stress intensity factor k_1 leads to $k_1 = pa^{\beta}\sin(\pi\beta)(\pi\beta)^{-1}$, which reproduces the known closed-form result for the singular stress field around a penny-shaped crack in an axially graded medium [6]. A distinctive feature of a non-homogeneous body is that the order of the stress singularity is determined by the gradient parameter n; as a result, the stress distribution near the boundary differs qualitatively from that in a homogeneous solid.

The healing effect is governed by a single parameter — the product $\varepsilon\lambda$ — so filler stiffness and interlayer thinness are interchangeable: a compliant but very thin layer is mechanically equivalent to a stiffer, thicker one. Even an extremely compliant filler removes most of the stress concentration, whereas further increases in stiffness or reductions in thickness yield increasingly smaller effects. Material gradation increases the stress intensity level in an unhealed crack; however, the relative effectiveness of healing increases with the degree of gradation, since a gradient body is more compliant near the crack plane; therefore, healing by injection in a functionally graded material proves to be more, rather than less, effective than in a homogeneous material. For ultra-thin interlayers the residual stress concentration is a small fraction of that of the open crack, which confirms the efficiency of injection strengthening of crack-damaged nonhomogeneous bodies.

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В.П. Силованюк, https://orcid.org/0000-0002-3765-5962
А.І. Дідух, https://orcid.org/0009-0005-2265-7411

Фізико-механічний інститут ім. Г.В. Карпенка НАН України, Львів, Україна
E-mail: dandriy066@gmail.com

ЗАЛІКОВУВАННЯ ДИСКОПОДІБНОЇ ТРІЩИНИ У ФУНКЦІОНАЛЬНО-ГРАДІЄНТНОМУ МАТЕРІАЛІ

Композити зі змінними за неперервним законом об'ємними частками називають функціонально-градієнтними матеріалами. Для функціонально-градієнтних матеріалів досліджено вплив неперервної осьової градації пружних властивостей та тонкого податливого заліковувального прошарку на дископодібну тріщину. Тіло моделюється як ізотропне середовище, модуль зсуву якого змінюється за степеневим законом уздовж осі, перпендикулярної до площини тріщини, тоді як коефіцієнт Пуассона залишається сталим. Затверділий ін'єкційний наповнювач заповнює порожнину тріщини у вигляді тонкого шару змінної товщини і передає зусилля між берегами тріщини відповідно до контактної моделі Вінклера. Застосування інтегральних перетворень Ганкеля разом із модифікованою прошарком умовою на берегах тріщини зводить задачу до інтегрального рівняння Фредгольма другого роду відносно розкриття тріщини. Рівняння розв'язано чисельно квадратурним методом і обчислено нормований коефіцієнт інтенсивності напружень як функцію параметра градації, жорсткості наповнювача та відносної товщини прошарку. Аналіз показує, що розв'язок залежить від наповнювача лише через ефективну жорсткість, яка поєднує ці два параметри, тому жорсткість наповнювача і товщина прошарку є взаємозамінними: що тоншим є прошарок, то податливішого наповнювача достатньо для досягнення того самого ефекту залікування. Числові результати показують, що ефект залікування є різко нелінійним: навіть дуже податливий наповнювач знімає більшу частину концентрації напружень поблизу контуру тріщини, тоді як подальше збільшення жорсткості наповнювача чи зменшення товщини прошарку дає дедалі менший ефект. Для дуже тонких прошарків залишкова концентрація напружень знижується до малої частки від рівня незаповненої тріщини. Градієнтність матеріалу підвищує рівень інтенсивності напружень біля незалікованої тріщини, проте відносна ефективність заліковування зростає зі збільшенням ступеня градієнтності: за того самого наповнювача залишковий коефіцієнт інтенсивності напружень у сильно градієнтному тілі зменшується до меншої частки свого значення для незалікованої тріщини, ніж в однорідному тілі, тож ін'єкційне заліковування виявляється ще ефективнішим у функціонально-градієнтних матеріалах. Отримані результати обґрунтовують доцільність ін'єкційного зміцнення пошкоджених тріщинами елементів конструкцій із неперервно неоднорідних матеріалів.

Ключові слова: *тріщина, функціонально-градієнтний матеріал, заліковування тріщини, коефіцієнт інтенсивності напружень.*