doi: https://doi.org/10.15407/dopovidi2019.04.017
UDC 512.544

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On the structure of groups, whose subgroups are either normal or core-free

Presented by Corresponding Member of the NAS of Ukraine V.P. Motornyi

We investigate the influence of some natural types of subgroups on the structure of groups. A subgroup H of the group G is called core-free if $\mathrm{Core}_G(H) = \langle 1 \rangle$*. We study the groups, in which every subgroup is either normal or core-free. More precisely, we obtain the structures of monolithic and non-monolithic groups with this property.*

Keywords: *normal subgroup, core-free subgroup, Dedekind group.*

Let G be a group. The following two normal subgroups are associated with any subgroup H of the group G: H^G, the normal closure of H in a the group G, the least normal subgroup of G including H, and $\mathrm{Core}_G(H)$, the (normal) core of H in G, the greatest normal subgroup of G, which is contained in H. We have

$$H^G = \langle H^x \mid x \in G \rangle$$

and

$$\mathrm{Core}_G(H) = \bigcap_{x \in G} H^x .$$

A subgroup H is normal if and only if $H = H^G = \mathrm{Core}_G(H)$. In this sense, the subgroups H, for which $\mathrm{Core}_G(H) = \langle 1 \rangle$, are the complete opposite of the normal subgroups. A subgroup H of the group G is called *core-free* in G if $\mathrm{Core}_G(H) = \langle 1 \rangle$.

There is a whole series of papers devoted to the study of groups with only two types of subgroups: subgroups with some property ρ and subgroups with a property that is antagonistic to ρ (see, for example, [1—6]). In particular, from the results of paper [3], it is possible to obtain a description of groups that have only two possibilities for each subgroup H: $H^G = H$ or $H^G = G$. In this connection, a dual question naturally arises on the structure of groups, in which, for each subgroup H, there are only two other possibilities: $\mathrm{Core}_G(H) = H$ or $\mathrm{Core}_G(H) = \langle 1 \rangle$. The finite groups having this property had been studied in [7]. Note at once that the groups, whose all subgroups are normal, possess this property.

Recall that a group G is called *Dedekind*, if every its subgroup is normal. The Dedekind group G has the following structure: it is either Abelian or $G = Q_8 \times D \times P$, where Q_8 is a quaternion group of order 8, D is an elementary Abelian 2-group, and P is an Abelian 2′-group [8].

Another extreme case that occurs here is the simple groups. In them, every proper subgroup is core-free. This fact immediately shows that the study of groups, in which $\mathrm{Core}_G(H) = H$ or $\mathrm{Core}_G(H) = \langle 1 \rangle$ for each subgroup H, makes sense for generalized soluble groups. The two key cases here are as follows: G is a non-monolithic group or G is a monolithic group. Let G be a group. The intersection of all non-trivial normal subgroups $\mathrm{Mon}(G)$ of G is called the *monolith* of the group G. If $\mathrm{Mon}(G) \neq \langle 1 \rangle$, then the group G is called *monolithic*, and, in this case, $\mathrm{Mon}(G)$ is the least non-trivial normal subgroup of G.

Our first main result is related to the non-monolithic case.

Theorem A. *Let G be an infinite group, whose non-normal subgroups are core-free. If G is non-monolithic, then G is a Dedekind group.*

The following our main theorem considers the monolithic case. Here, we get a much more diverse situation. Separate considerations are required for non-periodic and periodic groups.

Theorem B. *Let G be a locally soluble non-periodic group, whose non-normal subgroups are core-free. Suppose that G is not a Dedekind group. Then G is monolithic, the factor-group $G/\mathrm{Mon}(G)$ is non-periodic, $G = \mathrm{Mon}(G) \leftthreetimes A$, and the following conditions hold:*

(*i*) $\mathrm{Mon}(G)$ *is either torsion-free Abelian subgroup or elementary Abelian p-subgroup for some prime p;*

(*ii*) $[G, G] = \mathrm{Mon}(G) = C_G(\mathrm{Mon}(G))$;

(*iii*) *a subgroup A is Abelian, and* $\mathrm{Tor}(A)$ *is locally cyclic;*

(*iv*) *if* $\mathrm{Mon}(G)$ *is an elementary Abelian p-subgroup, then* $\mathrm{Tor}(A)$ *is a p'-subgroup;*

(*v*) *if A has finite* 0*-rank, then* $\mathrm{Mon}(G)$ *is an elementary Abelian p-subgroup;*

(*vi*) *if B is another complement to* $\mathrm{Mon}(G)$ *in G, then the subgroups A and B are conjugate.*

In turn, the case where G is periodic also splits into two cases depending on whether the center includes a monolith or not. Recall that a p-group G is called *extraspecial*, if $[G, G] = \zeta(G)$ is a subgroup of order p and $G/\zeta(G)$ is an elementary Abelian p-group.

From this definition, we can see that the center of an extraspecial p-group G is the least normal subgroup, so that if H is a subgroup of G, and H includes a non-trivial G-invariant subgroup, then H includes $\zeta(G)$. The equality $[G, G] = \zeta(G)$ implies that H is normal in G. In other words, every subgroup of G is either normal or core-free.

Theorem C. *Let G be a periodic monolithic group, whose non-normal subgroups are core-free. Suppose that G is not a Dedekind group. If the center of G includes a monolith, then $G = KE$, where K is a cyclic or quasicyclic p-subgroup, E is an extraspecial p-subgroup, $K = \zeta(G)$, and $K \cap E = [G, G]$ is a subgroup of order p, p is a prime.*

Theorem D. *Let G be an infinite periodic locally soluble monolithic group, whose non-normal subgroups are core-free. Suppose that G is not a Dedekind group and the monolith of G is not central. Then $G = \mathrm{Mon}(G) \leftthreetimes A$, and the following conditions hold:*

(*i*) $\mathrm{Mon}(G)$ *is an infinite elementary Abelian p-subgroup for some prime p, and A is an infinite periodic p'-group;*

(*ii*) $[G, G] = \mathrm{Mon}(G) = C_G(\mathrm{Mon}(G))$;

(*iii*) *whether the subgroup A is locally cyclic, or* $A = Q \times B$ *, where Q is a quaternion group of order* 8*, and B is a locally cyclic* 2′*-subgroup;*

(*iv*) *if C is another complement to* $\mathrm{Mon}(G)$ *in G, then the subgroups A and C are conjugate.*

Note that if $G/\mathrm{Mon}(G)$ is finite or $\mathrm{Mon}(G)$ is finite and non-central, then *G* is finite (this follows from Theorem D). The last our result gives a description of the finite soluble group, whose non-normal subgroups are core-free. As was noted above, a finite group, whose non-normal subgroups are core-free, was studied in [7]. Our description is more detailed than the description given in Theorem 1 of that paper. We also note that the proof of Lemma 5 in [7] contains a gap (only the case where the both factor-groups G/N_1 and G/N_2 are non-Abelian was considered). In addition, there is a mistake there: the fact that *H* is a subgroup of $T \times A$ does not implies that $H = H_1 \times H_2$, where $H_1 \leqslant T$ and $H_2 \leqslant A$. Therefore, we do not use the results of work [7]. We proved of the following result.

Theorem E. *Let G be a finite soluble group, whose non-normal subgroups are core-free. Suppose that G is not a Dedekind group. Then G is monolithic.*

If the center of G includes a monolith, then $G = KE$ *where K is a cyclic p-subgroup, E is an extraspecial p-subgroup,* $K = \zeta(G)$*, and* $K \cap E = [G, G]$ *is a subgroup of order p, p is a prime.*

If the monolith of G is not central, then $G = \mathrm{Mon}(G) \rtimes A$*, and the following conditions hold:*

(*i*) $\mathrm{Mon}(G)$ *is elementary Abelian p-subgroup for some prime p, and A is a p′-group;*

(*ii*) $[G, G] = \mathrm{Mon}(G) = C_G(\mathrm{Mon}(G))$*;*

(*iii*) *whether a subgroup A is cyclic or* $A = Q \times B$*, where Q is a quaternion group of order* 8*, and B is a cyclic* 2′*-subgroup;*

(*iv*) *if C is another complement to* $\mathrm{Mon}(G)$ *in G, then the subgroups A and C are conjugate.*

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Received 02.01.2019

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ПРО СТРУКТУРУ ГРУП, ПІДГРУПИ ЯКИХ
АБО НОРМАЛЬНІ, АБО ВІЛЬНІ ВІД ЯДРА

Досліджується вплив деяких природних типів підгруп на структуру груп. Підгрупу H групи G називаємо вільною від ядра, якщо $\mathrm{Core}_G(H) = \langle 1 \rangle$. Вивчено групи, в яких кожна підгрупа або нормальна, або вільна від ядра. Точніше, одержано будову монолітичних та немонолітичних груп з цією властивістю.

Ключові слова: *нормальна підгрупа, вільна від ядра підгрупа, дедекіндова група.*

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О СТРУКТУРЕ ГРУПП, ПОДГРУППЫ КОТОРЫХ
ЛИБО НОРМАЛЬНЫ, ЛИБО СВОБОДНЫ ОТ ЯДРА

Исследуется влияние некоторых естественных типов подгрупп на структуру групп. Подгруппу H группы G называем свободной от ядра, если $\mathrm{Core}_G(H) = \langle 1 \rangle$. Изучены группы, в которых каждая подгруппа либо нормальна, либо свободна от ядра. Точнее, получена структура монолитических и немонолитических групп с этим свойством.

Ключевые слова: *нормальная подгруппа, свободная от ядра подгруппа, дедекиндова группа.*