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Necessary condition for the existence of a simple closed geodesic on a regular tetrahedron in the spherical space

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In the spherical space the curvature of the tetrahedron's faces equals 1, and the curvature of the whole tetrahedron is concentrated into its vertices and faces. The intrinsic geometry of this tetrahedron depends on the value α *of faces angle, where* $\pi/3 < \alpha \leqslant 2\pi/3$. *The simple (without points of self-intersection) closed geodesic has the type (p,q) on a tetrahedron, if this geodesic has p points on each of two opposite edges of the tetrahedron, q points on each of another two opposite edges, and (p+q) points on each edges of the third pair of opposite one. For any coprime integers (p,q), we present the number* $\alpha_{p,\,q}$ $(\pi/3 < \alpha_{p,\,q} < 2\pi/3)$ *such that, on a regular tetrahedron in the spherical space with the faces angle of value* $\alpha > \alpha_{p,\,q}$, *there is no simple closed geodesic of type (p,q)*

Ключові слова: *closed geodesic, regular tetrahedron, spherical space.*

In 1905 H. Poincaré conjectured that, on a smooth convex surface in three-dimensional Euclidean space, there exists a simple (without points of self-intersection) closed geodesic. In 1929, L. Lyusternik and L. Shnirelman proved that at least three simple closed geodesics exist on a compact simply-connected two-dimensional Riemannian manifold [1, 2]. It is natural to consider properties of geodesic lines in spaces of negative and positive curvatures. From H. Huber's works [3, 4] it is known that, on a complete hyperbolic surface (two-dimensional Riemannian manifold of constant curvature -1) of finite area, the number of all closed geodesics of length not greater than L is of order e^L/L as L tends to infinity. I. Rivin and M. Mirzakhani proved that, on a hyperbolic surface of genus g with n points at infinity (cusps), the number of simple closed geodesics of length bounded above by L is asymptotic to the constant times $L^{6g-6+2n}$, when L goes to infinity [5, 6]. From the results of V. Toponogov [7], it follows that, on a complete simple-connected regular two-dimensional Riemannian manifold of Gaussian curvature $\geqslant k > 0$, a simple closed geodesic has the length not greater than $2\pi/\sqrt{k}$. V. Vaigant and O. Matukevich

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proved that, on such surface a geodesic line of length not less than $3\pi/\sqrt{k}$ has a point of self-intersection [8].

Properties of geodesics on the non-smooth surfaces, especially on convex polyhedra, are also investigated. A *geodesic* is a curve such that any sufficiently small subarc of this curve realizes the shortest path between endpoints of this subarc. On a convex polyhedron, the geodesic is realized by the straight line segment within any face, the geodesic does not pass through the vertices of the polyhedron, and the geodesic forms the equal angles with an edge in the adjacent faces.

D. Fuchs and K. Fuchs supplemented and systemized the results about closed geodesics on regular polyhedra in three-dimensional Euclidean space [9, 10]. V. Protasov described the structure of simple closed geodesics on a simplex in three-dimensional Euclidean space and found the estimation the number of these geodesics depending on the greatest deviation from π of the sum of the face's angles at the vertices of a simplex [11].

A pair of coprime integers (p,q) is called a *type* of a simple closed geodesic on a tetrahedron, if this geodesic has p points on each of two opposite edges of the tetrahedron, q points on each of another two opposite edges, and $(p+q)$ points on each edges of the third pair of opposite one.

It is known that, on a regular tetrahedron in Euclidean space, any closed geodesic has no points of self-intersection, and it is uniquely characterized by the pair of coprime integers (p,q). Moreover, for any coprime integers (p,q), there exist infinitely many simple closed geodesics of type (p,q) on the tetrahedron such that their segments are parallel to each other within a tetrahedron's face. The length of the closed geodesic of type (p,q) on the tetrahedron with the edges of value 1 is equal to

$$L = 2\sqrt{p^2 + pq + q^2}\,. \tag{1}$$

In work [12], simple closed geodesics on regular tetrahedra in hyperbolic space was studied. In this space the curvature of the tetrahedron's faces is equal to -1, so the curvature of the tetrahedron concentrates both on its vertices and its faces. The intrinsic geometry of this tetrahedron depends on the value α of its face's angle, that satisfies the inequality $0 < \alpha < \pi/3$. It is proved that on a regular tetrahedron in hyperbolic space for any coprime integers (p,q), $0 \leqslant p < q$, there exists unique, up to the rigid motion of the tetrahedron, simple closed geodesic of type (p,q). These geodesics exhaust all simple closed geodesics on a regular tetrahedron in hyperbolic space. Furthermore, on this tetrahedron, the number of simple closed geodesics of length bounded by L is asymptotic to constant (depending on α) times L^2, when L tends to infinity.

In this work we consider regular tetrahedra in the spherical space. In this case, the curvature of a tetrahedron's face is equal to 1, and the curvature of the tetrahedron also concentrates on its vertices and its faces. The intrinsic geometry of a tetrahedron in the spherical space depends on the value α of the face's angle and $\pi/3 < \alpha \leqslant 2\pi/3$. The length a of the tetrahedron's edge is equal to

$$a = \arccos\left(\frac{\cos\alpha}{1-\cos\alpha}\right). \tag{2}$$

If $\alpha = 2\pi/3$, then the tetrahedron coincides with the unit two-dimensional sphere. Hence, there are infinitely many simple closed geodesics on it, and they are the great circles of the sphere. In the following, we consider that $\pi/3 < \alpha < 2\pi/3$.

Lemma 1. *The length of a simple closed geodesic on a regular tetrahedron in the spherical space is less than* 2π.

It is possible to prove Lemma 1 exploring the structure of a simple closed geodesic on the regular tetrahedron and considering the special development of the tetrahedron along this geodesic on the unit two-dimensional sphere. However, Lemma 1 could be considered as the particular case of the result proved by Alexander A. Borisenko [13] about the generalization of the Toponogov theorem [14] to the case of two-dimensional Alexandrov space.

We prove the following statement.

Theorem 1. *Let* (p,q) *be a pair of coprime integers. If the value* α *satisfies the inequality*

$$\alpha > 2\arcsin\sqrt{\frac{p^2+pq+q^2}{4(p^2+pq+q^2)-\pi^2}}, \tag{3}$$

then, on a regular tetrahedron with face's angle α *in spherical space there is no simple closed geodesic of type* (p,q).

Proof. Let $A_1A_2A_3A_4$ be a regular tetrahedron in the spherical space with the face's angle α, and let γ be a simple closed geodesic of type (p,q) on it. A tetrahedron's face is isometric to the convex regular triangle bounded by the shortest geodesic arcs on a unit two-dimensional sphere. Consider the unit sphere containing the face $A_1A_2A_3$. Construct the Euclidean plane Π passing through the vertices A_1, A_2 and A_3. The intersection of the sphere with Π is a small circle. Build a ray starting at the sphere's center O and passing through a point at the triangle $A_1A_2A_3$. This ray intersects the plane Π, so we get the geodesic map between the sphere and the plane Π. The image of the spherical triangle $A_1A_2A_3$ is the triangle $\Delta A_1A_2A_3$ at the Euclidean plane Π. The edges of $\Delta A_1A_2A_3$ are the chords joining the vertices of the spherical triangle. From (2), it follows that the length $\tilde{a}$ of the plane triangle's edge equals

$$\tilde{a} = \frac{\sqrt{4\sin^2\frac{\alpha}{2}-1}}{\sin\frac{\alpha}{2}}. \tag{4}$$

The segments of the geodesic γ lying inside $A_1A_2A_3$ are mapped into the straight line segments inside $\Delta A_1A_2A_3$.

In the similar way, the other tetrahedron faces $A_2A_3A_4$, $A_2A_4A_1$, and $A_1A_3A_4$ are mapped into the plane triangles $\Delta A_2A_3A_4$, $\Delta A_2A_4A_1$, and $\Delta A_1A_3A_4$. Since the spherical tetrahedron is regular, the constructed plane triangles are equal. We can glue them together identifying the edges with the same labels. Hence, we obtain the regular tetrahedron in the Euclidean space. Since the segments of γ are mapped into the straight line segments within the plane triangles, then these straight line segments form the broken line $\tilde{\gamma}$ on the regular Euclidean tetrahedron, and $\tilde{\gamma}$ passes through the tetrahedron's edges at the same order as the simple closed geodesic of type (p,q).

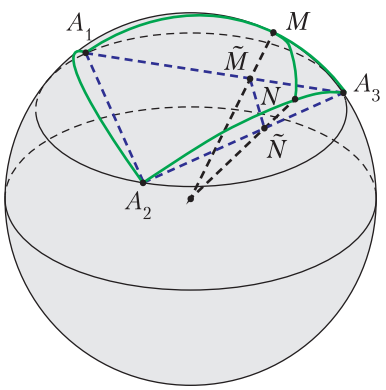

Fig. 1

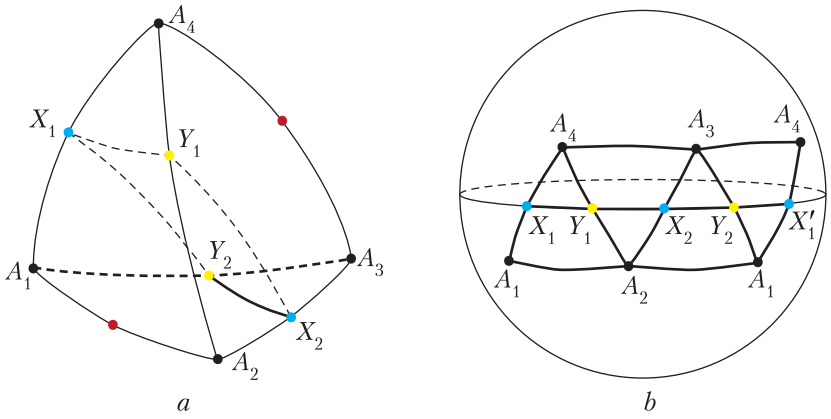

Fig. 2

Let us show that the length of γ is greater than the length of $\tilde{\gamma}$. Consider an arc MN of the geodesic γ within the face $A_1A_2A_3$. The segments OM, ON intersect the plane Π at the points $\tilde{M}$, $\tilde{N}$, respectively. The straight line segment $\tilde{M}\tilde{N}$ lying into $\Delta A_1A_2A_3$ is the image of the arc MN under the geodesic map (Fig. 1). Suppose that the length of the arc MN is equal to 2ϕ. Then the length of the segment $\tilde{M}\tilde{N}$ is equal to $2\sin\phi$. Thus, the length of any arc of γ is greater than the length of its image on $\tilde{\gamma}$.

Consider the development of the Euclidean tetrahedron along the broken line $\tilde{\gamma}$. Since $\tilde{\gamma}$ passes through the tetrahedron's edges at the same order as the simple closed geodesic of type (p,q). Then there is a straight line segment inside this development that corresponds to the simple closed geodesic μ of type (p,q) on this tetrahedron. From (1) and (4), it follow that the length of μ is equal to

$$L_\mu = 2\sqrt{p^2+pq+q^2}\,\frac{\sqrt{4\sin^2\dfrac{\alpha}{2}-1}}{\sin\dfrac{\alpha}{2}}.$$

Thus, we obtain that the length of the simple closed geodesic γ on the regular tetrahedron in the spherical space is greater than the length of the broken line $\tilde{\gamma}$ on the regular tetrahedron in the Euclidean space, which is not less than the length of the simple closed geodesic μ on the same Euclidean tetrahedron. Hence, the length of γ is greater than L_μ.

From Lemma 1, it follows that if α fulfils the inequality

$$2\sqrt{p^2+pq+q^2}\,\frac{\sqrt{4\sin^2\dfrac{\alpha}{2}-1}}{\sin\dfrac{\alpha}{2}} > 2\pi, \tag{5}$$

then the necessary condition for the existence of the simple closed geodesic of type (p,q) on a regular tetrahedron with face's angle α in spherical space is failed.

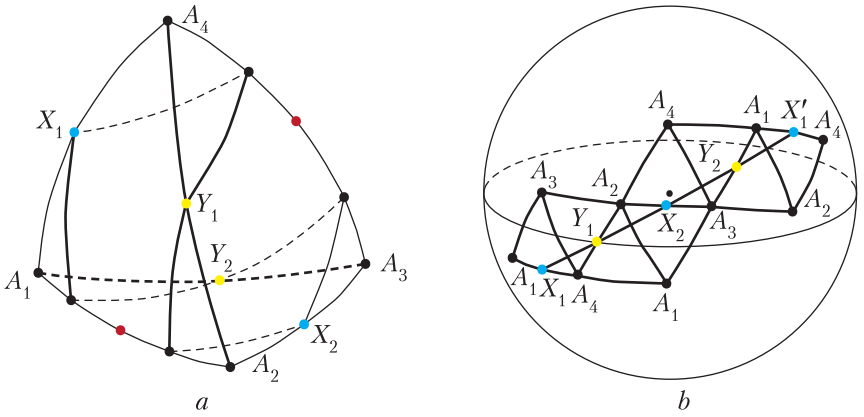

Fig. 3

Hence, after modifying (5), we get that if

$$\alpha > 2\arcsin\sqrt{\frac{p^2+pq+q^2}{4(p^2+pq+q^2)-\pi^2}},$$

then there is no simple closed geodesic of type (p,q) on the tetrahedron with face's angle α in spherical space. Theorem 1 is proved.

Corollary 1. *On a regular tetrahedron in the spherical space, there exist the finite number of simple closed geodesics.*

If the numbers (p,q) tends to infinity, then

$$\lim_{p,q\to\infty} 2\arcsin\sqrt{\frac{p^2+pq+q^2}{4(p^2+pq+q^2)-\pi^2}} = \frac{\pi}{3}.$$

From (3), it follows that a simple closed geodesic of type (p,q), where (p,q) are large, could exist on a regular tetrahedron with face's angle close to $\pi/3$.

In addition, consider two particular cases.

Lemma 2. *On a regular tetrahedron with the face's angle* α *such that* $\pi/3 < \alpha < 2\pi/3$, *there exists three simple closed geodesics of type* (0,1).

These geodesics coincide up to the rigid motion of the tetrahedron.

A geodesic of type (0,1) consists of four segments connecting the midpoints of four edges sequentially and doesn't pass through a pair of opposite edges of the tetrahedron (Fig. 2).

Moreover, if the value of face's angle of the regular tetrahedron satisfies $\pi/2 \leqslant \alpha < 2\pi/3$, then this tetrahedron has only three geodesics of type (0,1) and doesn't have other ones.

Lemma 3. *On a regular tetrahedron with the face's angle* α *such that* $\pi/3 < \alpha < \pi/2$, *there exist three simple closed geodesics of type* (1,1).

These geodesics also coincide up to the rigid motion of the tetrahedron (Fig. 3).

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REFERENCES

1. Lusternik, L. A. & Schnirelmann, L. G. (1927). On the problem of tree geodesics on the surfaces of genus zero. C. R. Acad. Sci., 189, pp. 269-271 (in French).
2. Lusternik, L. A. & Schnirelmann, L. G. (1947). Topological methods in variational problems and their application to the differential geometry of surfaces. Uspekhi Mat. Nauk, 2, Iss. 1, pp. 166-217 (in Russian).
3. Huber, H. (1959). On the analytic theory hyperbolic spatial forms and motion groups. Math. Ann., 138, pp.1-26 (in German). https://doi.org/10.1007/BF01369663
4. Huber, H. (1961). On the analytic theory hyperbolic spatial forms and motion groups II. Math. Ann., 143, pp. 463-464 (in German). https://doi.org/10.1007/BF01451031
5. Rivin, I. (2001). Simple curves on surfaces. Geometriae Dedicata, 87, pp. 345-360. https://doi.org/10.1023/A:1012010721583
6. [6] Mirzakhani, M. (2008). Growth of the number of simple closed geodesics on hyperbolic surfaces. Ann. Math., 168, pp. 97-125. https://doi.org/10.4007/annals.2008.168.97
7. Toponogov, V. A. (1959). Riemann spaces with curvature bounded below. Uspekhi Mat. Nauk, 14, Iss. 1, pp. 87-130 (in Russian).
8. Vaigant, V. A. & Matukevich, O. Yu. (2001). Estimation of the length of a simple geodesic on a convex surface. Siberian Math. J., 42, No. 5, pp. 833-845. https://doi.org/10.1023/A:1011951207751
9. Fuchs, D. & Fuchs, E. (2007). Closed geodesics on regular polyhedra. Mosc. Math. J., 7, No. 2, pp. 265-279. https://doi.org/10.17323/1609-4514-2007-7-2-265-279
10. Fuchs, D. (2009). Geodesics on a regular dodecahedron: Preprints of Max Planck Institute for Mathematics, Bonn, 91, pp. 1-14. Retrieved from http://webdoc.sub.gwdg.de/ebook/serien/e/mpi_mathematik/2010/2009_91.pdf
11. Protasov, V. Yu. (2007). Closed geodesics on the surface of a simplex. Sb. Math., 198, No. 2, pp. 243-260. https://doi.org/10.4213/sm1501
12. Borisenko, A. A. & Sukhorebska, D. D. (2020). Simple closed geodesics on regular tetrahedra in Lobachevsky space. Sb. Math., 211, No. 5, pp. 617-642. https://doi.org/10.1070/sm9212
13. Borisenko, A. A. (2020). The estimation of the length of a convex curve in two-dimensional Alexandrov space. J. Math. Phys., Anal., Geom., 16, No. 3.
14. Toponogov, V. A. (1963). Estimation of the length of a convex curve on a two-dimensional surface. Sibirsk. Mat. Zhurn., 4, No. 5, pp. 1189-1183 (in Russian).

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НЕОБХІДНА УМОВА ІСНУВАННЯ ПРОСТОЇ ЗАМКНЕНОЇ ГЕОДЕЗИЧНОЇ НА ПРАВИЛЬНОМУ ТЕТРАЕДРІ У СФЕРИЧНОМУ ПРОСТОРІ

У сферичному просторі кривина граней тетраедра дорівнює 1, і кривина усього тетраедра зосереджена як у його вершинах, так і на гранях. Внутрішня геометрія правильного тетраедра у сферичному просторі залежить від величини α кута його грані, де $\pi/3 < \alpha \leqslant 2\pi/3$. Проста (без самоперетину) замкнена геодезична на тетраедрі має тип (p,q), якщо ця геодезична перетинає у p точках одну пару протилежних ребер тетраедра, у q точках — іншу пару протилежних ребер тетраедра і у $(p+q)$ точках — третю пару протилежних ребер тетраедра. Показано, що для кожної пари взаємно простих натуральних чисел (p,q) існує таке число $\alpha_{p,q}$ ($\pi/3 < \alpha_{p,q} < 2\pi/3$), що на правильному тетраедрі у сферичному просторі з кутом грані величини $\alpha > \alpha_{p,q}$ не існує простої замкненої геодезичної типу (p,q).

Ключові слова: *замкнені геодезичні, правильний тетраедр, сферичний простір.*