ДОПОВІДІ
НАЦІОНАЛЬНОЇ
АКАДЕМІЇ НАУК
УКРАЇНИ

https://doi.org/10.15407/dopovidi2025.02.003
UDC 517.5

V.Ya. Gutlyanskiĭ[1, 2], https://orcid.org/0000-0002-8691-4617
V.I. Ryazanov [1, 2], https://orcid.org/0000-0002-4503-4939
A.R. Salimov[1], https://orcid.org/0009-0000-1021-7527
R.R. Salimov[2], https://orcid.org/0000-0001-9395-3334

[1] Institute of Applied Mathematics and Mechanics of the NAS of Ukraine, Slovyansk, Ukraine
[2] Institute of Mathematics of the NAS of Ukraine, Kyiv, Ukraine
E-mail: vgutlyanskii@gmail.com, vl.ryazanov1@gmail.com, salimov.artem@gmail.com, ruslan.salimov1@gmail.com

On the Dirichlet problem for generalized Cauchy-Riemann equations

Presented by Academician of the NAS of Ukraine I.I. Skrypnik

Here we give a survey of consequences of the theory of the Beltrami equations from the complex analysis for the Dirichlet problem to generalized Cauchy-Riemann equations $\nabla v = B\nabla u$ in the real plane R^2 that describe flows of fluids in anisotropic and inhomogeneous media, where B is a 2×2 matrix valued coefficient and the gradients ∇u and ∇v are interpreted as vector columns. Moreover, we clarify the relationships of the latter to the A-harmonic equation $div\,(A\nabla u) = 0$ with matrix valued coefficients A that is one of the main equations of the potential theory, namely, of the hydromechanics (fluid mechanics) in anisotropic and inhomogeneous media in the plane. The survey includes a series of effective integral criteria for existence of regular solutions of the Dirichlet problem with continuous data in arbitrary bounded simple connected domains to generalized Cauchy-Riemann equations with matrix coefficients in the case of anisotropic and inhomogeneous media.

Keywords: *Cauchy-Riemann system, generalized Cauchy-Riemann equations, Dirichlet problem, Beltrami and A-harmonic equations.*

1. Introduction. As it is well-known, the characteristic property of an analytic function $f = u + iv$ in the complex plane C is that its real and imaginary parts satisfy the **Cauchy-Riemann system**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \quad z = x + iy\,. \tag{1}$$

Euler was the first who has found the connection of the system (1) with the analytic functions.

Citation: Gutlyanskiĭ V.Ya., Ryazanov V.I., Salimov A.R., Salimov R.R. On the Dirichlet problem for generalized Cauchy—Riemann equations. *Dopov. nac. akad. nauk Ukr.* 2025. No 2. P. 3—10. https://doi.org/10.15407/dopovidi2025.02.003

A physical interpretation of (1), going back to Riemann works on function theory, is that u represents a **potential function** of the incompressible fluid steady flow in homogeneous isotropic media and v is its **stream function**.

This system can be written as the one equation in the matrix form

$$\nabla v = H\nabla u, \tag{2}$$

where ∇v and ∇u denotes the gradient of v and u, correspondingly, interpreted as vector-columns in R^2, and $H : R^2 \to R^2$ is the so-called **Hodge operator** represented as the 2×2 matrix

$$H = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{3}$$

which carries out the counterclockwise rotation of vector columns by the angle $\pi/2$ in R^2.

Thus, (2) shows that streamlines and equipotential lines of the fluid flow are mutually orthogonal. Note also that H is an analog of the imaginary unit in the space $M^{2\times 2}$ of all 2×2 matrices with real entries because

$$H^2 = -I, \tag{4}$$

where I is the unit 2×2 matrix.

Here we consider the **generalized Cauchy-Riemann equations** of the form

$$\nabla v = B\nabla u \tag{5}$$

with the matrix valued coefficients $B : D \to M^{2\times 2}$ that describe flows in anisotropic and inhomogeneous media and, on the basis of the well-developed theory of the Beltrami equations, see e.g. monographs [1]—[6] and articles [7]—[9], we give the corresponding consequences for the Dirichlet problem with continuous data to these equations.

Moreover, let us clarify the relationships of the equations (5) and the A **-harmonic equation**

$$div(A(Z)\, grad\ u(Z)) = 0, \quad Z := (x, y) \in R^2, \tag{6}$$

with matrix valued coefficients $A : D \to M^{2\times 2}$ that is one of the main equations of hydromechanics in anisotropic and inhomogeneous media.

For this purpose, recall that the Hodge operator H transforms curl-free fields into divergence-free fields and vice versa. Thus, if $u \in W^{1,1}_{loc}$ is a solution of (6) in the sense of distributions, then the field $V = HA\nabla u$ is curl-free and, consequently, $V = \nabla v$ for some $v \in W^{1,1}_{loc}$ and the pair (u, v) is a solution of the equation (6) in the sense of distributions with

$$B := H\cdot A. \tag{7}$$

Vice versa, if u and $v \in W^{1,1}_{loc}$ satisfies (5) in the sense of distributions, then u satisfies (6) also in the sense of distributions with

$$A := -H\cdot B = H^{-1}\cdot B \tag{8}$$

because the curl of any gradient field is zero in the sense of distributions.

Let us denote by $B^{2\times 2}$ space of all 2×2 matrices with real entries,

$$B=\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \tag{9}$$

with $\det B=1$, antisymmetric with respect to its auxiliary diagonal, i.e., with $b_{22}=-b_{11}$, and with the **ellipticity condition** $|\mu|<1$, where

$$\mu=\mu_B:=\frac{b_{12}+b_{21}-2ib_{11}}{b_{12}-b_{21}-2}. \tag{10}$$

Note that, under the above conditions $\det B=1$ and $b_{22}=-b_{11}$, the ellipticity condition $|\mu|<1$ is equivalent to the condition $b_{21}>b_{12}$ and, furthermore, to the conditions $b_{12}<0$ and $b_{21}>0$.

We will call the quantity (10) a complex characteristic of (5). Criteria of solvability of the equation (5) will be formulated in terms of its dilatation quotient

$$K_{\mu_B}:=\frac{1+|\mu_B|}{1-|\mu_B|}. \tag{11}$$

Let us consider the Dirichlet problem for the generalized Cauchy-Riemann equations (5) consisting in finding its solutions (u,v) with prescribed continuous data $\phi:\partial D\to R$ of potential u at the boundary

$$\lim_{z\to\zeta} u(z)=\phi(\varsigma)\quad \forall\varsigma\in\partial D \tag{12}$$

in arbitrary bounded simply connected domains D in R^2.

Given a simply connected domain D in R^2, we say that a pair (u,v) of continuous functions $u:D\to R$ and $v:D\to R$ in the class $W^{1,1}_{loc}$ is a **regular solution of the Dirichlet problem** (12) for the generalized Cauchy-Riemann equation (5) in D if (u,v) satisfies (5) a.e. in D and, moreover, $\nabla u\neq 0$ and $\nabla v\neq 0$ a.e. in D, and the correspondence $(x,y)\mapsto(u,v)$ is a discrete and open mapping of D into R^2. Recall that a mapping of a domain D in R^2 into R^2 is called **discrete** if the preimage of each point in R^2 consists of isolated points in D and **open** if the mapping maps every open set in D onto an open set in R^2.

2. Criteria in terms of BMO, FMO and VMO. Hereafter $dL(Z)$ corresponds to the Lebesgue measure in R^2 with the notation $Z:=(x,y)\in R^2$.

Recall first of all that a real-valued function Φ in a domain D of R^2 is called of **bounded mean oscillation** in D, abbr. $\Phi\in BMO(D)$, if

$$\|\Phi\|_*=\sup_B\frac{1}{|B|}\int_B|\Phi(Z)-\Phi_B|\,dL(Z)<\infty, \tag{13}$$

where $\Phi\in L^1_{loc}(D)$, the supremum is taken over all discs B in D and

$$\Phi_B:=\frac{1}{|B|}\int_B\Phi(Z)\,dL(Z).$$

We also write $\Phi \in BMO(\overline{D})$ if $\Phi_* \in BMO(D_*)$ for some extension Φ_* of the function Φ into a domain D_* containing $\overline{D}$.

The class BMO was introduced by John and Nirenberg (1961) in the paper [10] and soon became an important concept in harmonic analysis, partial differential equations and related areas, see e.g. monographs [11] and [12].

Further we always assume by definition that $K_{\mu_B} \equiv 1$ outside D.

Theorem 1. *Let D be a bounded simply connected domain in R^2 and let $B: D \to B^{2\times 2}$ be a measurable function. Suppose also that K_{μ_B} has a dominant $Q: R^2 \to [1, \infty)$ in the class $\Phi \in BMO(\overline{D})$. Then the generalized Cauchy-Riemann equation (5) has a regular solution (u, v) of the Dirichlet problem (12) in D for each continuous inconstant boundary date $\phi: \partial D \to R$.*

A function Φ in BMO is said to have **vanishing mean oscillation**, abbr. $\Phi \in VMO(D)$, if the supremum in (13) taken over all balls B in D with $|B| < \varepsilon$ converges to 0 as $\varepsilon \to 0$. VMO has been introduced by Sarason in [13]. There are a number of papers devoted to the study of PDEs with coefficients of the class VMO. Note that $W^{1,2}(D) \subset VMO(D)$, see e.g. [14].

Corollary 1. *In particular, the conclusion of Theorem 1 on existence of a regular solution for the Dirichlet problem (12) to the generalized Cauchy-Riemann equation (5) holds if the dominant Q of K_{μ_B} belongs to the class $W^{1,2}(\overline{D})$.*

Following [15], we say that a locally integrable function $\Phi: D \to R$ **has finite mean oscillation** at a point $Z_0 \in D$, abbr. $\Phi \in FMO(Z_0)$, if

$$\overline{\lim_{\varepsilon \to 0}} \frac{1}{|B(Z_0, \varepsilon)|} \int_{B(Z_0, \varepsilon)} \left|\Phi(Z) - \tilde{\Phi}_{\varepsilon}(Z_0)\right| dL(Z) < \infty, \tag{14}$$

where

$$\tilde{\Phi}_{\varepsilon}(Z_0) = \frac{1}{|B(Z_0, \varepsilon)|} \int_{B(Z_0, \varepsilon)} \Phi(Z) dL(Z) < \infty, \tag{15}$$

is the mean integral value of the function $\Phi(Z)$ over disk $B(Z_0, \varepsilon) := \{Z \in R^2 : |Z - Z_0| < \varepsilon\}$.

Theorem 2. *Let D be a bounded simply connected domain in R^2 and $B: D \to B^{2\times 2}$ be a measurable function in D. Suppose also that $K_{\mu_B}(Z) \leqslant Q_{Z_0}(Z)$ a.e. in U_{Z_0} for every point $Z_0 \in \overline{D}$, a neighbourhood U_{Z_0} of Z_0 and a function $Q_{Z_0}(Z): U_{Z_0} \to [0, \infty]$ in the class $FMO(Z_0)$. Then the generalized Cauchy-Riemann equation (5) has a regular solution (u, v) of the Dirichlet problem (12) in D for each continuous inconstant boundary date $\phi: \partial D \to R$.*

Corollary 2. *Let D be a bounded simply connected domain in R^2 and $B: D \to B^{2\times 2}$ be a measurable function in D. Suppose also that*

$$\overline{\lim_{\varepsilon \to 0}} \frac{1}{|B(Z_0, \varepsilon)|} \int_{B(Z_0, \varepsilon)} K_{\mu_B}(Z)\, dL(Z) < \infty \qquad \forall\, Z_0 \in \overline{D} \tag{16}$$

Then the conclusion of Theorem 2 holds.

Corollary 3. *Let D be a bounded simply connected domain in R^2 and $B: D \to B^{2\times 2}$ be a measurable function. Suppose also that $K_{\mu_B}(Z) \leqslant Q_{Z_0}(Z)$ a.e. in D with a function Q of the class $FMO(\overline{D})$. Then the conclusion holds.*

3. Criteria of the Calderon-Zygmund type.

Theorem 3. *Let* D *be a bounded simply connected domain in* R^2 *and* $B: D \to B^{2\times 2}$ *be a measurable function. Suppose also that*

$$\int_{\varepsilon<|Z-Z_0|<\varepsilon_0} K_{\mu_B}(Z)\,\frac{dL(Z)}{|Z-Z_0|^2} = o([\log(1/\varepsilon)]^2) \qquad \forall\, Z_0 \in \overline{D} \tag{17}$$

as $\varepsilon \to 0$ *for some* $\varepsilon_0 = \varepsilon(Z_0) > 0$. *Then the generalized Cauchy-Riemann equation (5) has a regular solution* (u, v) *of the Dirichlet problem (12) in* D *for each continuous inconstant boundary date* $\phi: \partial D \to R$.

Remark 1. We are also able here to replace (17) by

$$\int_{\varepsilon<|Z-Z_0|<\varepsilon_0} \frac{K_{\mu_B}(Z)\,dL(Z)}{\left(|Z-Z_0|\cdot\log\left(\frac{1}{|Z\_Z_0|}\right)\right)^2} = o([\log\log(1/\varepsilon)]^2) \qquad \forall\, Z_0 \in \overline{D} \tag{18}$$

In general, we are able to give here the whole scale of the corresponding conditions in terms of iterated logarithms, i.e., in terms of functions of the form $1/\left(t\log 1/t\cdot\log\log 1/t\cdot\ldots\cdot\log\ldots\log 1/t\right)$.

4. Criteria of the Lehto type. Further $k_{\mu_B}(Z_0, r)$ denotes the integral mean of $K_{\mu_B}(Z)$ over the circle $S(Z_0, r) := \{Z \in R^2 : |Z - Z_0| = r\}$.

Theorem 4. *Let* D *be a bounded simply connected domain in* R^2 *and* $B: D \to B^{2\times 2}$ *be a measurable function with* $K_{\mu_B} \in L^1(D)$. *Suppose also that, for some* $\varepsilon_0 = \varepsilon(Z_0) > 0$,

$$\int_0^{\varepsilon_0} \frac{dr}{r k_{\mu_B}(Z_0, r)} = \infty \quad \forall\, Z_0 \in \overline{D}. \tag{19}$$

Then the generalized Cauchy-Riemann equation (5) has a regular solution (u, v) *of the Dirichlet problem (12) in* D *for each continuous inconstant boundary date* $\phi: \partial D \to R$.

Corollary 4. *Let* D *be a bounded simply connected domain in* R^2 *and* $B: D \to B^{2\times 2}$ *be a measurable function in* D *with* $K_{\mu_B} \in L^1(D)$. *Suppose also that*

$$k_{\mu_B}(Z_0, \varepsilon) = O\left(\log\frac{1}{\varepsilon}\right) \ \textit{as}\ \varepsilon \to 0 \quad \forall\, Z_0 \in \overline{D}. \tag{20}$$

Then the conclusion of Theorem 4 holds.

Remark 2. In particular, the conclusion of Theorem 4 holds if

$$K_{\mu_B}(Z) = O\left(\log\frac{1}{|Z-Z_0|}\right) \ \text{as}\ Z \to Z_0 \quad \forall\, Z_0 \in \overline{D}. \tag{21}$$

Moreover, the condition (20) can be replaced by the whole series of more weak conditions

$$k_{\mu_B}(Z_0, \varepsilon) = O\left(\left[\log\frac{1}{\varepsilon}\cdot\log\log\frac{1}{\varepsilon}\cdot\ldots\cdot\log\ldots\log\frac{1}{\varepsilon}\right]\right) \ \text{as}\ \varepsilon \to 0 \quad \forall\, Z_0 \in \overline{D}. \tag{22}$$

5. Criteria of the Orlicz type.

Theorem 5. *Let* D *be a bounded simply connected domain in* R^2 *and* $B : D \to B^{2\times 2}$ *be a measurable function. Suppose also that*

$$\int_D \Phi(K_{\mu_B}(Z))\, dL(Z) < \infty. \tag{23}$$

for a convex non-decreasing function $\Phi : [0, \infty] \to [0, \infty]$ *such that, for some* $\Delta > 0$,

$$\int_\Delta^\infty \log \Phi(t) \frac{dt}{t^2} = +\infty. \tag{24}$$

Then the generalized Cauchy-Riemann equation (5) has a regular solution $(u, \ v)$ *of the Dirichlet problem (12) in* D *for each continuous inconstant boundary date* $\phi : \partial D \to R$.

Remark 3. Note that the condition (24) is not only sufficient but also necessary to have regular solutions (u, v) of the Dirichlet problem (12) in D to the generalized Cauchy-Riemann equations (5) with the integral constraints (23) for all continuous inconstant date $\phi : \partial D \to R$.

Corollary 5. *Let* D *be a bounded simply connected domain in* R^2 *and* $B : D \to B^{2\times 2}$ *be a measurable function. Suppose that, for* $\alpha > 0$,

$$\int_D \exp[\alpha K_{\mu_B}(Z)]\, dL(Z) < \infty. \tag{25}$$

Then the conclusion of Theorem 5 holds.

The corresponding survey of consequences on the Dirichlet problem to generalized Cauchy-Riemann equations with sources from the theory of the Beltrami equations will be published elsewhere.

Acknowledgments. *The first 2 authors are partially supported by the project "Mathematical modeling of complex dynamical systems and processes caused by the state security", No. 0123U100853, of the National Academy of Sciences of Ukraine and by a grant from the Simons Foundation PD-Ukraine-00010584.*

REFERENCES

1. Ahlfors, L. (1966). Lectures on Quasiconformal Mappings. New York: Van Nostrand.
2. Astala, K., Iwaniec, T. & Martin, G. J. (2009). Elliptic differential equations and quasiconformal mappings in the plane. Princeton Math., Ser. 48. Princeton: Princeton Univ. Press.
3. Bojarski, B., Gutlyanskii, V., Martio, O. & Ryazanov, V. (2013). Infinitesimal geometry of quasiconformal and bi-Lipschitz mappings in the plane. EMS Tracts in Mathematics, 19. Zürich: European Mathematical Society.
4. Gutlyanskii, V., Ryazanov, V., Srebro, U. & Yakubov, E. (2012). The Beltrami Equation: A Geometric Approach. Developments in Mathematics 26. Berlin: Springer.
5. Lehto, O. & Virtanen, K. I. (1973). Quasiconformal mappings in the plane. Die Grundlehren der mathematischen Wissenschaften 126. Berlin-Heidelberg-New York: Springer.
6. Martio, O., Ryazanov, V., Srebro, U. & Yakubov, E. (2009). Moduli in modern mapping theory. Springer Monographs in Mathematics. New York: Springer.
7. Gutlyanskii, V., Martio, O., Sugawa, T. & Vuorinen, M. (2005). On the degenerate Beltrami equation. Trans. Amer. Math. Soc., 357(3), pp. 875-900.
8. Gutlyanskii, V., Nesmelova, O., Ryazanov, V. & Yakubov, E. (2024). BMO and Dirichlet problem for semi-linear equations in the plane. Topological Methods in Nonlinear Analysis, 64(1), pp. 163-199. https://doi.org/10.12775/TMNA.2023.052.
9. Gutlyanskii, V., Ryazanov, V., Sevost'yanov, E. & Yakubov, E. (2022). BMO and Dirichlet problem for degenerate Beltrami equation. J. Math. Sci. (New York), 268(2), pp. 157-177. (Translation from Ukr. Mat. Visn., 19(3), pp. 327-354).
10. John, F. & Nirenberg, L. (1961). On functions of bounded mean oscillation. Comm. Pure Appl. Math., 14, pp. 415-426.
11. Heinonen, J., Kilpeläinen, T. & Martio, O. (1993). Nonlinear A-harmonic theory of degenerate elliptic equations. Oxford Mathematical Monographs. Oxford Science Publications, The Clarendon Press. New York: Oxford Univ. Press.
12. Reimann, H. M. & Rychener, T. (1975). Funktionen Beschränkter Mittlerer Oscillation. Lecture Notes in Math.
13. Sarason, D. (1975). Functions of vanishing mean oscillation. Trans. Amer. Math. Soc., 207, pp. 391-405.
14. Brezis, H. & Nirenberg, L. (1995). Degree theory and BMO. I. Compact manifolds without boundaries. Selecta Math. (N.S.), 1(2), pp. 197-263.
15. Ignat'ev, A. A. & Ryazanov, V. I. (2005). Finite mean oscillation in the mapping theory. Ukrainian Math. Bull., 2(3), pp. 403-424.

Received 18.01.2025

В.Я. Гутлянський[1, 2], https://orcid.org/0000-0002-8691-4617
В.І. Рязанов[1, 2], https://orcid.org/0000-0002-4503-4939
А.Р. Салімов[1], https://orcid.org/0009-0000-1021-7527
Р.Р. Салімов[2], https://orcid.org/0000-0001-9395-3334

[1] Інститут прикладної математики і механіки НАН України, Слов'янськ, Україна
[2] Інститут математики НАН України, Київ, Україна
E-mail: vgutlyanskii@gmail.com, vl.ryazanov1@gmail.com,
salimov.artem@gmail.com, ruslan.salimov1@gmail.com

ПРО ЗАДАЧУ ДІРІХЛЕ ДЛЯ УЗАГАЛЬНЕНИХ РІВНЯНЬ КОШІ—РІМАНА

Стаття містить огляд наслідків теорії рівнянь Бельтрамі з комплексного аналізу для задачі Діріхле до узагальненого рівняння Коші—Рімана $\nabla v = B\nabla u$ на дійсній площині R^2, що описує потоки рідини в анізотропних та неоднорідних середовищах, де коефіцієнт B представлено у вигляді 2×2 матриці, а градієнти ∇u та ∇v інтерпретуються як вектор-стовпці. Крім того, з'ясовується зв'язок цього рівняння з A-гармонічним рівнянням $div(A\nabla u) = 0$ з матричними коефіцієнтами A, яке є одним із головних рівнянь теорії потенціалу, а саме гідромеханіки (механіки рідин) в анізотропних та неоднорідних середовищах на площині. Огляд включає низку ефективних інтегральних критеріїв існування регулярних розв'язків задачі Діріхле з неперервними даними в довільних обмежених однозв'язних областях для узагальнених рівнянь Коші—Рімана з матричними коефіцієнтами в умовах анізотропних та неоднорідних середовищ.

Ключові слова: *система Коші—Рімана, узагальнені рівняння Коші—Рімана, задача Діріхле, рівняння Бельтрамі, A-гармонічні рівняння.*